Cognitive Psychology 78 (2015) 28–56

Contents lists available at ScienceDirect

Cognitive Psychology

journal homepage: www.elsevier.com/locate/cogpsych

Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes

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ARTICLE INFO

Article history:
Accepted 28 January 2015

Keywords:
Fractions
Distance effects
Number sense
Numerical representations

ABSTRACT

This study presents evidence that humans have intuitive, perceptually based access to the abstract fraction magnitudes instantiated by nonsymbolic ratio stimuli. Moreover, it shows these perceptually accessed magnitudes can be easily compared with symbolically represented fractions. In cross-format comparisons, participants picked the larger of two ratios. Ratios were presented either symbolically as fractions or nonsymbolically as paired dot arrays or as paired circles. Response patterns were consistent with participants comparing specific analog fractional magnitudes independently of the particular formats in which they were presented. These results pose a challenge to accounts that argue human cognitive architecture is ill-suited for processing fractions. Instead, it seems that humans can process nonsymbolic ratio magnitudes via perceptual routes and without recourse to conscious symbolic algorithms, analogous to the processing of whole number magnitudes. These findings have important implications for theories regarding the nature of human number sense – they imply that fractions may in some sense be natural numbers, too.

1. Introduction

Formal number concepts and the mathematics built upon them were invented too recently to have influenced the evolution of our species (Dehaene & Cohen, 2007). How is it then that evolutionarily

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http://dx.doi.org/10.1016/j.cogpsych.2015.01.006

ancient human brains can support these relatively recent numerical inventions? To address this question, researchers often look to the basic cognitive architectures upon which culturally established number concepts might be built. The counting numbers (i.e., 1, 2, 3...) – which mathematicians have dubbed 'natural' numbers – are often the focal point of these theories. It makes intuitive sense that these 'natural' numbers might form the groundwork of our understanding of mathematics. These numbers play a major role not just in counting, but in numerical cognition more generally (Butterworth, 2010; Gerstmann, 1940; Noël, 2005). Moreover, they map onto basic perceptual abilities that enumerate discrete sets. This ability to perceptually estimate discrete numerical magnitudes – an ability granted by what is known as the approximate number system (ANS) – is present not only in humans but across multiple species (e.g., Dehaene, Dehaene-Lambertz, & Cohen, 1998; Meck & Church, 1983). Indeed, several researchers have argued that the acquisition of abstract numerical concepts rests upon these evolutionarily inherited enumeration abilities (e.g., Dehaene, 1997; Feigenson, Dehaene, & Spelke, 2004; Nieder, 2005; Piazza, 2010). By positing such a crucial role for perceptually based enumeration in the development of number concepts, these theories privilege natural numbers by proxy, essentially echoing Kroenecker's famous dictum that "God made the integers; all the rest is the work of man" (Bell, 1986, p. 477).

However, this ostensibly obvious intuition may obscure the possibility that natural numbers and enumeration are not alone in their 'naturalness'. In this study, we administered cross-format comparison tasks to explore whether humans have an intuitive sense of nonsymbolic ratio magnitude that allows them to perceive and judge fractional[1] number values in ways similar to how the approximate number system allows them to perceive and judge natural number magnitudes. The cross format nature of the comparisons is important: Successful comparison within a particular format might be accomplished by methods that need not necessarily require magnitude abstraction, such as scaling (e.g., Ahl, Moore, & Dixon, 1992).

By contrast, cross-format comparisons require some sort of abstraction of magnitudes to allow comparison on the same scale. Each comparison involved fraction magnitudes instantiated in nonsymbolic forms that were not amenable to simple enumeration or to manipulation via symbolic algorithms, insuring any such abstractions must be perceptually based.

Minimally, two pieces of evidence seem important to support the possibility that participants perceive abstract ratio magnitudes:

1. Participants must prove sensitive to the equivalence of fraction values across formats when perception is the only plausible route to identifying those magnitudes, and
2. Participants must complete comparisons in a short enough time course to preclude the use of conscious algorithms.

Patterns consistent with these constraints would suggest that this sense of proportion is unlikely to be dependent upon enumeration or estimation of natural number values. In short, such results would imply that fractions are in some sense 'natural' numbers too.

1.1. Primitive ratio processing – A link to natural fraction concepts?

The proposition that fractional number values may be intuitive might seem at odds with the fact that both children and even highly educated adults often experience considerable difficulties understanding symbolic fractions (e.g., Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981; Ni & Zhou, 2005; Siegler & Pyke, 2012). For instance, when a nationally representative sample of children was asked whether 12/13 + 7/8 was closest to 1, 2, 19, or 21, 8th-graders chose 19 and 21 more often than 2 (Carpenter et al., 1981). These problems extend well past middle school, persisting into adulthood. On the same estimation problem, a nationally representative sample of 17-yr-olds was correct only 37% of the time. Moreover, Stigler, Givvin, and Thompson (2010) found that only 33% of their sample of community college students could accurately find the largest of four simple fractions. Many have

[1] Although we note that a mathematically rigorous treatment of the terms 'fraction' and 'ratio' considers ratio to be one of several possible interpretations of fraction concepts, we will use the terms interchangeably throughout this manuscript.

argued that these well documented difficulties with fractions stem from innate constraints on human cognitive architectures (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Dehaene, 1997; Feigenson et al., 2004; Gallistel & Gelman, 1992; Geary, 2007; Gelman & Williams, 1998; Wynn, 1990). Dehaene (1997) cogently encapsulated the core of such *innate constraints accounts* when he wrote:

> Some mathematical objects now seem very intuitive only because their structure is well adapted to our brain architecture. On the other hand, a great many children find fractions very difficult to learn because their cortical machinery resists such a counterintuitive concept (p. 7).

According to innate constraints theorists, whole number abilities are supported by perceptual systems that evolved to process discrete numerosities (i.e., sets of countable objects), and these systems serve as evolutionary precursors for supporting understanding of symbolic numbers (Bailey, Hoard, Nugent, & Geary, 2012; Dehaene & Cohen, 2007; Feigenson et al., 2004; Gallistel & Gelman, 1992; Nieder, 2005; Piazza, 2010). Such accounts contend that the major cognitive module for processing numbers, the ANS, is fundamentally designed to deal with discrete numerosities that correspond to whole number values. Therefore, innate constraints theorists argue, fractions and rational number concepts are difficult because they *lack* a similarly intuitive basis and must instead be built from systems originally developed to support whole number understanding.

However, Siegler and colleagues have recently called into question the practice of treating fractional values solely as educational constructs, suggesting that researchers should reexamine the nature of fractional quantities in hopes of developing a more integrated theory of numerical understanding that is inclusive of both natural numbers and fractions (Fazio, Bailey, Thompson, & Siegler, 2014; Siegler, Fazio, Bailey, & Zhou, 2013). The present work addresses this issue. By focusing on perceptual abilities that naturally map onto fraction magnitudes, this research may show rational numbers to be on more equal footing relative to natural numbers. Contrary to the innate constraints perspective, we argue that human cognitive architecture is very much compatible with fraction concepts.

A growing body of evidence suggests that an intuitive, perceptually based cognitive system for processing nonsymbolically instantiated fractional magnitudes may indeed exist (e.g., Duffy, Huttenlocher, & Levine, 2005; Jacob & Nieder, 2009b; Jacob, Vallentin, & Nieder, 2012; McCrink & Wynn, 2007). This cognitive system seems to represent and process magnitudes of nonsymbolic ratios (hereafter also referred to as nonsymbolic fractions) in several representational formats, such as fractions formed by the relative lengths of two lines. Several extant lines of research suggest that this sensitivity to nonsymbolic ratio magnitudes may emerge before formal education and that it even extends across species (e.g., Jacob et al., 2012; McCrink & Wynn, 2007).

McCrink and Wynn (2007) investigated this ratio perception from a developmental perspective. They habituated 6-month old infants to specific nonsymbolic ratios instantiated in multiple versions using blue dots and yellow Pac-Men (e.g. the ratio 2:1 was instantiated using multiple sets, including the ratios 8:4, 38:19, and 22:11). After habituation, infants looked longer at novel ratio stimuli that differed by a factor of two (e.g., a 4:1 ratio). McCrink and Wynn thus revealed a specific ability of infants to perceive differences in ratios (i.e. nonsymbolic fractions) as opposed to the overall number of items in a set. Moreover, some studies with preschool-aged children suggest that the perception and encoding of the ratio between two stimuli may actually be easier for young children than encoding absolute values of a single stimulus (Duffy et al., 2005; Huttenlocher, Duffy, & Levine, 2002). Other work has similarly demonstrated human sensitivity to nonsymbolic fraction magnitudes across the developmental time span (Boyer & Levine, 2012; Jacob & Nieder, 2009b; Jacob et al., 2012; Mix, Levine, & Huttenlocher, 1999; Singer-Freeman & Goswami, 2001; Sophian, 2000; Sophian & Wood, 1997; Spinillo & Bryant, 1991).

This sensitivity to nonsymbolic fraction magnitudes does not end with *homo sapiens*. Recent research has also demonstrated that non-human primates can accurately assess the magnitudes of ratios composed of pairs of non-symbolic magnitudes. Vallentin and Nieder (2008) trained monkeys on match-to-sample tasks using ratios composed of pairs of line segments (e.g., one half instantiated as ___ or as ___). The monkeys performed far better than chance (85.5% accuracy), showing considerable sensitivity to specific fractional magnitudes and even rivaling adult human performance on the same task. Moreover, using single-celled recordings from the monkeys, Vallentin and Nieder also found individual neurons that responded to specific ratio values constructed of line segments.

These neurons fired strongly in response to particular ratio magnitudes, without regard to the sizes of their components.

In sum, studies have found the ability to process nonsymbolic ratios in non-human primates (Vallentin & Nieder, 2008), 6-month old infants (McCrink & Wynn, 2007), elementary school aged children (Jeong, Levine, & Huttenlocher, 2007) and adults (Fabbri, Caviola, Tang, Zorzi, & Butterworth, 2012; Meert, Grégoire, Seron, & Noël, 2012). The existence of this sensitivity among uneducated children and even across species led Jacob et al. (2012) to posit that dedicated neural networks have evolved that automatically process nonsymbolic ratio magnitudes. The current research tests one implication of the existence of such a dedicated ratio processing system: If human beings possess a system that allows them to represent fractional magnitudes abstractly and amodally, it follows that they should be able (a) to process and compare nonsymbolic fraction magnitudes instantiated in different formats and (b) to complete such processing perceptually, without recourse to symbolic algorithms.

1.2. Numerical distance effects and nonsymbolic fractions

Despite the clear evidence that humans can perceive nonsymbolic ratio magnitudes, the question of whether symbolic fractions and nonsymbolic fractions map to the same analog ratio magnitude code remains unexplored. To date, there is little evidence demonstrating a link between perceptual sensitivity to nonsymbolic ratio magnitudes (which exists even among innumerate infants and non-human primates) and the *acquired* understanding for magnitudes of symbolic fractions (but see Fazio et al., 2014; Matthews, Chesney, & McNeil, 2014). Previous studies of ratio estimation and production have required some level of cross-format mapping of fraction magnitudes (Barth & Paladino, 2011; Hollands & Dyre, 2000; Spence, 1990; Stevens & Galanter, 1957; Varey, Mellers, & Birnbaum, 1990) but were not explicitly designed to encourage rapid perceptual access of the holistic magnitudes instantiated by nonsymbolic ratios. These studies have generally used estimation or scaling paradigms that unfortunately precluded analysis of perceived similarity when stimuli of different magnitudes are compared – precisely the type of analysis that is usually conducted to assess intuitive analog representations of magnitude (Halberda & Feigenson, 2008; Meert, Grégoire, & Noël, 2010; Moyer & Landauer, 1967; Schneider & Siegler, 2010). Consequently, questions of whether maps between symbolic and nonsymbolic fractions access the same amodal magnitude remain unexplored.

We aimed to fill this gap by asking if cross-format comparisons of fractions instantiated in multiple formats (i.e. perceptually accessed ratios composed of dots or circles vs. traditional Arabic fraction symbols) demonstrate *numerical distance effects*. The numerical distance effect – the phenomenon whereby error rates and reaction times vary negatively with increasing distance between the magnitudes of stimuli to be compared – is considered to be a hallmark of analog magnitude representation (Moyer, Bradley, Sorensen, Whiting, & Mansfield, 1978; Nieder, 2005; Schneider & Siegler, 2010; Sekuler & Mierkiewicz, 1977; Siegler & Pyke, 2012; Sprute & Temple, 2011). The numerical distance effect as originally conceived by Moyer and Landauer (1967) was seen as a special case of a more general process of magnitude comparison. Moyer and Landauer argued that if number comparisons proceeded on a perceptual level, then numerical comparisons should follow the same sort of psychophysical functions as "judgments of inequality for length of lines, pitch and colour..." Following Moyer and Landauer's precedent, the existence of distance effects among numerical stimuli has generally been taken to indicate an intuitive representation of the magnitudes of a given class of numbers on an internal mental number line (Dehaene, Dupoux, & Mehler, 1990; Kallai & Tzelgov, 2009; Nieder, 2005; Restle, 1970; Rubinsten, Henik, Berger, & Shahar-Shalev, 2002; Schneider & Siegler, 2010).

Until recently, numerical distance effects were primarily investigated using symbolic whole numbers or their nonsymbolic analogs – numerosities. However, several recent studies have shown that human adults: (a) exhibit distance effects when comparing symbolic fractions under certain conditions (DeWolf, Grounds, Bassok, & Holyoak, 2014; Jacob & Nieder, 2009a; Kallai & Tzelgov, 2009, 2012; Meert et al., 2010, 2012; Schneider & Siegler, 2010; Siegler, Thompson, & Schneider, 2011) and (b) exhibit distance effects when comparing nonsymbolic ratios within a particular format (Jacob & Nieder, 2009b; Vallentin & Nieder, 2008). Critically, we posit that *different* formats for fraction magnitudes, whether symbolic or nonsymbolic, converge on a single amodal representation of magnitude, (i.e., a 'mental number line'; Siegler et al., 2011). We therefore predict that participants should exhibit distance effects for cross-format comparisons of fractions similar to those typically found for within-format magnitude comparisons (Halberda & Feigenson, 2008; Moyer et al., 1978; Odic, Libertus, Feigenson, & Halberda, 2013). Such results would indicate that people indeed have intuitive perceptual access to abstract fractional magnitudes.

1.3. Overview of the current experiments

The experiments reported below investigated the existence of numerical distance effects using cross-format magnitude comparison tasks. Participants picked the larger of two stimuli, which were fractions instantiated either symbolically as Arabic numerals or nonsymbolically using stimuli whose magnitudes arguably could not be ascertained by use of conscious algorithms. In Experiment 1, we investigated whether participants would exhibit distance effects when comparing symbolic fractions to discrete but uncountable (in the brief time allotted) dot ratio stimuli. In Experiment 2, we investigated whether participants would exhibit distance effects when comparing symbolic fractions to continuous circle ratio stimuli whose individual components did not correspond to any particular number value. Finally, in Experiment 3 we investigated whether participants would exhibit distance effects when comparing dot ratio magnitudes to circle ratio magnitudes, two nonsymbolic forms.

2. Experiment 1

Experiment 1 was conducted to provide initial evidence that humans would show numerical distance effects when comparing fraction stimuli across representational formats. We used Arabic numerals and nonsymbolic dot arrays to present fractions of various magnitudes. The dot arrays were sufficiently numerous that they could not be serially counted in the time taken to complete trials and could not be easily partitioned. Thus, dot ratio magnitudes could not plausibly be ascertained via conscious algorithmic strategies. As a result, participants had to rely on perceptual routes to make judgments of these nonsymbolic fraction magnitudes. Numerical distance effects would suggest that (a) participants' perceptual systems were sensitive to fraction magnitudes when presented as nonsymbolic dot arrays and (b) these visuospatially accessed magnitudes can be mapped to symbolically represented magnitudes.

2.1. Method

2.1.1. Participants

Participants were 27 undergraduate students from a major Midwestern university, participating for partial course credit (22 female; ages $M = 19.7$, range = 18–22).

2.1.2. Materials and design

Participants completed paired comparisons in which they selected the larger of two fraction stimuli presented in two separate formats. One member of each pair was a symbolic fraction composed of Arabic numerals, and the other was a nonsymbolic ratio composed of dot arrays (see Fig. 1).

2.1.2.1. Symbolic stimuli. Symbolic stimuli were ten irreducible proper fractions composed of single digit Arabic numerators and denominators from 2 to 9 (i.e., 2/9, 2/7, 3/8, 3/7, 5/9, 3/5, 2/3, 3/4, 5/6, and 7/8, see Table 1). We chose fractions of this form because previous work has shown that irreducibility and exclusion of unit fractions (i.e. those with one in the denominator such as 1/9, 1/8, 1/7...) encourage participants to access the holistic magnitudes of fractions as opposed to using componential strategies (Kallai & Tzelgov, 2009, 2012; Meert et al., 2010; Schneider & Siegler, 2010; Sprute & Temple, 2011). The subset chosen spanned the range of 18 possible values for fractions of this type. Numerals for a given component were approximately 48.5 mm tall.

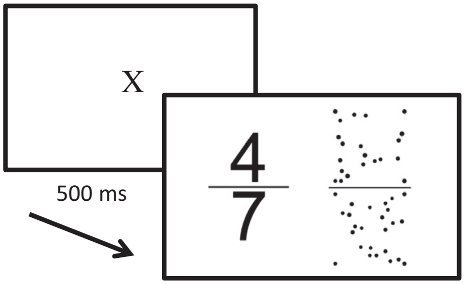

Fig. 1. Experiment 1: Sample depiction of a symbolic fraction vs. dot ratio comparison trial. Participants answered by pressing a button box indicating which stimulus they thought had the larger fractional value.

2.1.2.2. Dot ratio stimuli. Dot ratios were composed of pairs of dot arrays separated by a bar in the middle to form nonsymbolic ratios corresponding to the values 2/9, 2/7, 3/8, 3/7, 5/9, 3/5, 2/3, 3/4, 5/6, and 7/8 (e.g., the value of 2/9 might correspond to an array ratio of 33/150 dots). Although dot array ratios did not match these values exactly, all were within a 1% tolerance of the listed values. Arrays were composed of black dots on a white background. Displays controlled dot surface area so that all arrays had the same total surface area regardless of dot numerosity. Individual dot size varied both within and between arrays, such that the size of a given dot did not precisely correlate with array numerosity. Dots were randomly and evenly distributed in each array, and each nonsymbolic ratio occupied a 54×109 mm space.

A random number generator was used to create two separate sets of stimuli to control for correlation of component numerosity with total ratio magnitude (see Table 1). One set was constructed such that the numerosity of the numerator component was uncorrelated with overall ratio magnitude. A second set was constructed such that both denominator numerosity and total numerosity (i.e., numerator plus denominator) were uncorrelated with overall fraction magnitude. Component denominators varied considerably among stimuli within each set. When dot ratios were presented on the left side of a comparison, stimuli were selected from the set with uncorrelated denominator and total numerosity. When dot ratios appeared on the right, stimuli were selected from the set with uncorrelated numerators. To discourage participants from counting or use of computational procedures when estimating ratios, the smallest numerosity displayed in any given array was 21 (range of 21–150).

2.1.3. Procedure

Each participant completed all comparisons in one hour-long session that also included other comparison tasks not discussed in this report. Participants first saw instructions, then received five practice trials, and then performed the experimental trials. No feedback was given for practice or experimental trials.

For each trial, a fixation cross appeared in the center of the screen for 500 ms. The fixation cross was immediately followed by presentation of dot array and Arabic fraction comparison stimuli. The stimuli remained on-screen until the participant responded with a button box to indicate which fraction was greater (i.e., the left button to indicate the left side was greater, and right button to indicate the right was greater). For dot ratios, participants were specifically told that "the sizes of the dots don't matter. What matters is the ratio of the number of dots on top to the number of dots on the bottom" and that "There will be A LOT of dots for each ratio, so please don't try to count. Just go by feel." All 90 possible magnitude combinations were presented two times each – once with dots on the left and once with dots on the right. This resulted in a total of 180 trials per participant. Trial order was randomized, and each participant saw all the same stimuli.

2.1.4. Overview of analyses

As noted above, the numerical distance effect is a special case of a more general psychophysical function corresponding to the increased difficulty of discriminating between two magnitudes as the difference between them decreases. When comparing perceptually accessed stimulus magnitudes,

Table 1
Overall magnitudes and components of symbolic, dot, and circle ratio comparison stimuli in Experiments 1–3.

Stimulus format	Values									
Symbolic fractions	2/9	2/7	3/8	3/7	5/9	3/5	2/3	3/4	5/6	7/8
Dot ratios, uncorrelated numerators	33/150	33/116	27/72	21/49	26/47	22/37	47/71	31/41	21/25	46/53
Dot ratios, uncorrelated denominators	28/126	39/138	25/66	63/147	56/101	70/117	73/110	54/72	98/117	96/110
Circle ratios, uncorrelated numerators (mm/mm)	13.9/29.7	10.2/19.1	15.3/24.8	22.4/34.2	18.1/24.1	16.8/21.7	8.7/10.6	13.9/29.7	10.2/19.1	15.3/24.8
Circle ratios, uncorrelated denominators (mm/mm)	7.6/16.5	16.3/30.4	16.5/26.8	25.7/39.3	26.1/34.7	32.5/42.0	31.1/38.0	7.6/16.5	16.3/30.4	16.5/26.8
Decimal equivalents	.22	.29	.38	.43	.56	.6	.67	.75	.83	.88

Note: Dot ratios list the number of dots in numerator and denominator arrays. Circle ratios are listed terms of the diameters in mm of the component circles used. Area ratios correspond to the squares of the ratios formed by circle diameters.

people typically demonstrate an S-shaped sigmoid response curve that is symmetrical about a *point of subjective equality* (PSE). The PSE is the point at which participants on average judge presented stimuli as approximately equal in magnitude (i.e., the point at which participants were 50–50 on the larger vs. smaller judgment). Our initial analyses sought to map this psychophysical function, rather than following the more typical route of immediately regressing error rates or reaction latencies against the distance between stimuli (e.g., Dehaene et al., 1990; Kallai & Tzelgov, 2009, 2012; Moyer & Landauer, 1967).

It may be unreasonable to expect the PSE to converge exactly with the point of true equality when magnitudes are compared across different formats, as minor differences in stimuli can bias perceived magnitudes even within formats (e.g., the well known illusion that items higher relative to the horizon are perceived as larger, see Kingdom & Prins, 2009, for a discussion on how to analyze psychophysical data when such effects are present). Between-format magnitudes mappings can be both inexact and biased (e.g., Izard & Dehaene, 2008). Simply put, the point where stimuli *appear* to be equivalent need not agree with *objective* equality (Kingdom & Prins, 2009). This allows for a situation in which performance both demonstrates a perfectly symmetrical psychometric function, but simultaneously demonstrates a bias whereby the PSE is shifted from where it would canonically be expected to be. Use of simple error rates in such cases can obscure the distance-based symmetry of response patterns. Such biases are not necessarily expected in single-format studies where minor perceptual effects can be easily balanced, but must be tested and corrected before one can effectively use simple regression methods to investigate distance effects in cross-format studies.

With this in mind, our analyses always began with fitting participant responses to a psychophysical function. Specifically, we modeled participant response patterns using a logistic function, which is often used to model psychophysics functions (Kingdom & Prins, 2009; Wichmann & Hill, 2001).[2] Next, we estimated relevant PSEs as a measure of bias in converting magnitudes between formats. Finally, we performed a bias-corrected version of the traditional distance effect analyses that could appropriately assess the effects of interstimulus distance on accuracy and response latency in this cross-format design.

2.2. Results

2.2.1. The psychophysical function

Prior to analyses, responses with reaction times that were more than 3 SDs beyond the mean were removed. This resulted in a loss 1.9% of data points (94 of 4860). Preliminary inspection revealed that three participants appeared to have systematically reversed the instructions, consistently choosing the smaller stimulus for at least one presentation order. Two participants clearly chose the larger ratio when Arabic fractions were presented on the right, but chose the smaller ratio when Arabic fractions were presented on the left. The third participant reversed directions entirely, always selecting the smaller stimulus. This behavior was observed even for extreme distances characterized by low error rates in the overall sample (e.g. 2/9 vs. 7/8), indicating that it was not due to perceptual limitations, but instead reflected a systematic reversal of response choices. We excluded these three participants as non-compliant and ran our analysis on the remaining analytic sample of 24 participants. We only report results for the reduced sample, but follow-up analyses that included reverse-coded responses for the excluded participants yielded similar results.

We used logistic regression using all trials as data points with likelihood of judging the symbolic fraction stimulus to be larger as the dependent variable. The independent variables were: (1) *Distance*, calculated by subtracting the magnitude of the dot stimulus in each comparison pair from the magnitude of the symbolic stimulus, (2) *SymbolsRight*, a dummy-coded presentation order variable denoting whether symbolic fractions or nonsymbolic dot array fractions were presented on the right (SymbolsRight = 1), and (3) a *Distance* × *SymbolsRight* interaction term.

[2] Note, we chose to model the data using the logistic function – as opposed to other options such as the cumulative normal or the Weibull – for three reasons: (1) our primary purpose was to fit the data well and to estimate the PSE, not to make claims about the underlying neuropsychological mechanism of the judgment, (2) the logistic function is easily solved with relatively few data points, and (3) the logistic function is easily interpreted.

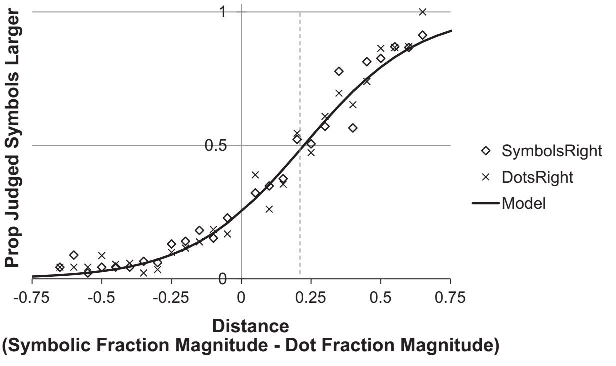

Fig. 2. Experiment 1: Proportion of trials in which participants judged the symbolic fraction to be larger than the nonsymbolic dot fraction as a function of the difference between stimuli magnitudes and whether the symbolic fraction or the dot fraction was presented on the right. Each plotted point represents the mean proportion of times respondents selected the symbolic fraction as greater. The solid curve represents estimated logistic fits to the data. Because there were no significant main effects or interactions based on presentation order, the model only predicts a single fit. The dotted vertical line marks the PSE of .21. *Note:* Distances were binned to the nearest .05 for illustrative purposes in the figure but were entered exactly into the regressions.

Table 2
Experiment 1, symbols vs. dots: results of logistic regression of probability of judging symbolic stimulus larger against distance and side of presentation – pooled and person-level data.

Parameter	Estimate
Pooled data: 4267 data points	
Nagelkerke Pseudo R^2	0.39
Constant (*SE*)	−1.07 (.06)[**]
Distance (*SE*)	4.98 (.23)[**]
SymbolsRight (*SE*)	0.12 (.09)
Distance × SymbolsRight (*SE*)	−0.37 (.32)
PSE	.21
Person level data	
Median Nagelkerke Pseudo R^2 (*SD*)	.54 (.17)
% of sample with significant effect for Distance	95.8% (23/24)
% of sample with significant effect for SymbolsRight	4.2% (1/24)
% of sample with significant effect for interaction	8.3% (2/24)
Mean PSE SymbolsRight (*SE*)[a]	.21 (.03)
Mean PSE DotsRight (*SE*)[a]	.22 (.03)

[**] $p < .01$.
[a] Means of individual PSEs were calculated after excluding the sole participant who failed to demonstrate a significant effect for distance. This participant had a SymbolsRight PSE of −.59, and a DotsRight PSE of 3.40.

The logistic function explained much of the variance, with Nagelkerke's Pseudo R^2 = .39. Participant judgments clearly exhibited a bias: As Fig. 2 shows, the collective sample showed a PSE of .21 regardless of whether symbolic fractions were displayed on the right or on the left, and thus regardless of whether stimuli were controlled such that the numerator or denominator was uncorrelated with overall ratio magnitude (see Table 2, top half). This indicates that throughout the range of comparisons, participants perceived dot ratios of a given magnitude as larger than their corresponding symbolic fractions by a value of ∼.21. For example, a dot ratio with a magnitude equivalent to .21 (or about 1/5) was seen as equivalent in size to a symbolic fraction with a magnitude of ∼.42 (about 3/7. Hence, the *perceived size* of a dot ratio could be estimated by adding .21 to the size of the

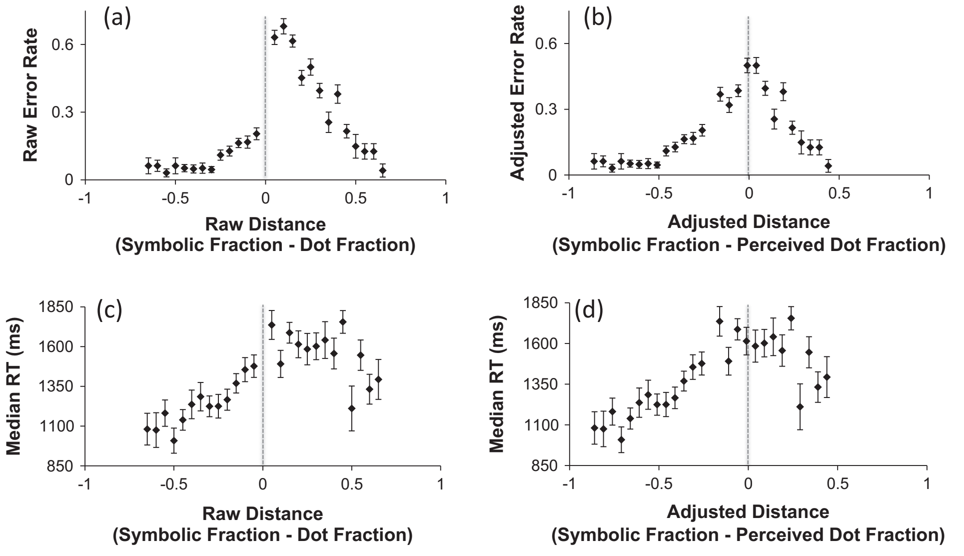

Fig. 3. Experiment 1: Error rate and RT patterns. (a) The graph of raw error rate displays a discontinuous jump due to the cross-format bias, with artificially high errors when symbolic stimuli are larger and artificially low errors when dot stimuli are larger. (b) Adjusted errors, calculated on the bias-corrected adjusted distance scale, are symmetrical about zero and reach a maximum of .5, regardless of the format of the larger stimulus. (c) Symmetry of the RT prior to bias correction is centered about approximately .2 to .3, confirming the bias of perceived equality. (d) By contrast, performance is centered closer to zero on the bias corrected distance scale. Error bars reflect standard errors of the means. *Note:* Distances were binned to the nearest .05 for illustrative purposes in the figure but were entered exactly into the regressions.

corresponding symbolic ratio. These results were not mere artifacts of pooling across participants. Logistic functions fit separately for each individual participant yielded patterns similar to the aggregate for 23 of 24 individuals (see Table 2).

2.2.2. Adjustments for bias

As anticipated, the subjective bias when comparing dot ratios and Arabic fraction values rendered raw accuracy inappropriate as a metric for examining how comparison performance varied with distance. The need to correct for bias is clearly demonstrated by the pattern observed in Fig. 3a: When calculated without corrections for bias, participant error rates rose only to 20% as distance approached zero when dot array ratios were the larger stimuli and discontinuously jumped to nearly 70% as this zero threshold was crossed and symbolic ratios became the larger stimuli. To illustrate the nature of this bias, consider the comparison of 5/9 (or .56) to the larger fraction 3/5 (or .6). When 5/9 is instantiated as dots, the perceptual bias amounts to comparing a dot ratio with a perceived magnitude of .77 (i.e., .56 + .21 due to bias) to a symbolic magnitude of .6, which would systematically generate an error. Similarly, when 3/5 is instantiated as dots, the bias effectively renders the comparison as .82 vs. .56, which would artificially decrease errors by exaggerating the difference in the correct direction.

However, we were able to create a bias adjusted metric that could parallel traditional numerical distance effect analyses and similarly capture the degree to which participant confidence falls as the absolute distance between stimulus magnitudes decreases. We applied adjustments to dot ratio magnitudes to correct for the cross-format bias in size perception. Because the PSE indicated dot ratios were perceived as .21 larger than symbolic fractions, we first added .21 to dot ratio magnitudes, yielding measures of *perceived magnitude*. Note that we could have alternatively chosen to subtract .21 from the magnitudes of symbolic fractions, as the process would have resulted in an equivalent translation of the distance between dot and symbolic stimuli. Next, we calculated *adjusted distance* and *adjusted accuracy* based on the new perceived magnitudes and their corresponding distances (defined for each comparison as symbolic magnitude – perceived dot ratio magnitude). The resulting adjusted error patterns were symmetrical about zero on the adjusted distance scale (see Fig. 3b) and were appropriate for applying standard regression analysis of distance effects. We performed a secondary analysis using reaction latencies as a face valid check on our bias adjustments, and this analysis yielded similar results (Fig. 3c and d).

2.2.3. Distance effects

To parallel analyses from previous research (Kallai & Tzelgov, 2009; Meert et al., 2012; Schneider & Siegler, 2010), we conducted analyses at two separate levels: group level regressions and supplemental individual level regressions. The group level analyses used the summary measures of mean adjusted error rates and median reaction times at each distance as the dependent variables. For individual level analyses, we performed regressions for each participant, using all individual trials as data points.

We regressed adjusted error rates and reaction times against: (1) *absolute adjusted distance*, calculated by taking the absolute value of adjusted distance as defined in Section 2.2.2, (2) *SymbolsRight*, as specified in Section 2.2.2, and (3) a *Distance* × *SymbolsRight* interaction term. Because some studies of the numerical distance effect have found that error falls off logarithmically with distance (e.g., Dehaene et al., 1990; Schneider & Siegler, 2010), we also ran parallel regressions using the logarithm of the adjusted distance as the independent variable in both error rate and RT regressions. Trials on which participants answered incorrectly (after adjustment) were excluded from the analysis of reaction times as is the standard procedure in the literature (Bonato et al., 2007; Dehaene et al., 1990; Kallai & Tzelgov, 2009, 2012; Meert et al., 2010; Sekuler & Mierkiewicz, 1977). Note that this bias correction resulted in adjusting distances of .21 to a corrected distance of 0. As items at a distance of 0 are equal, there is no "correct" answer. Thus, trials at this comparison distance (2% of the remaining data points) were dropped from the bias-corrected analyses.

2.2.3.1. Adjusted error rate analysis. The mean adjusted error rate was .23 ($SD = .42$) across all trials. Group level analysis (Table 3) revealed that error rates fell as distance increased ($\beta_{lin} = -.83$, $p < .01$; $\beta_{log} = -.85$, $p < .01$). Adjusted errors decreased from highs near 50% for comparisons with adjusted distances near zero to rates below 10% for comparisons over larger distances. Whether dot ratios were presented on right or left had no effect on accuracy ($\beta_{lin} = .04$, $p = .68$; $\beta_{log} = .06$, $p = .48$), nor did presentation position interact with distance ($\beta_{lin} = .01$, $p = .92$; $\beta_{log} = .02$, $p = .86$). All told, there was a pronounced effect of distance on accuracy, and the position of stimulus presentation did not moderate the effect.

Table 3
Experiment 1, symbols vs. dots: results of linear regression of error and response time against distance and side of presentation, pooled and person-level analyses.

Level of analysis, independent variables, and coefficients	Dependent variables			
	Linear regression		Logarithmic regression	
	Error rate	Median RT	Error rate	Median RT
Group level				
Adj. R^2	.68	.49	.70	.37
β Distance	−.83**	−.72**	−.85**	−.65**
β DotsLeft	.04	−.21	.06	−.10
β Distance × DotsLeft	.01	.06	.02	.09
Person level				
Median adj. R^2	.11	.05	.11	.05
SD of adj. R^2	.05	.06	.07	.05
% of sample with significant effect for Distance	91.7	58.3	91.7	45.8
	22/24	14/24	22/24	11/24
% of sample with significant effect for DotsLeft	0	0.0	0.0	4.2
	0/24	0/24	0/24	1/24
% of sample with significant effect for interaction	0.0	8.33	4.2	4.2
	0/24	2/24	1/24	1/24

Note. All coefficients are standardized.
** $p < .01$.

2.2.3.2. Reaction time analysis. Patterns with RT were consistent with those found for error rates (Table 3). It is noteworthy that the median RT was 1406 ms (SD = 1104) across all trials. This reaction latency was only ~300 ms longer than the ~1100 ms Halberda, Mazzocco, and Feigenson (2008) found participants took when comparing individual dot arrays in a single-format comparison task. This time difference is on the order of how long it takes to count a single item during rapid subvocal counting (i.e., about 240 ms, see Whalen, Gallistel, & Gelman, 1999). Given that any given arithmetic algorithm should take at least as long as subvocal counting, these RT data seem to preclude the possibility that participants used conscious algorithms for processing nonsymbolic ratios. If, as Halberda et al. (2008) attest, such fast reaction times indicate that participants were making comparisons based on perceptual judgments, the current data seem to warrant similar conclusions.

2.2.3.3. Diffusion modeling. The analyses presented so far examined RT and accuracy data separately. Moreover, the RT data from incorrect trials were excluded from the analyses. Recent work has shown that both these aspects of numeric discrimination tasks can be incorporated into a single analysis using a diffusion model (e.g., Ratcliff, Love, Thompson, & Opfer, 2012). We therefore used a diffusion model to establish whether our conclusions were supported when considering RTs and accuracy simultaneously. We subdivided the stimuli into 6 distance bins, with distance defined as the difference between the symbol ratio and the dot ratio (i.e., <−.33, −.33 to −.17, −.17 to 0, 0 to .17, .17 to .33, and >.33). Using Vandekerckhove and Tuerlinckx's (2007, 2008) Diffusion Model Analysis Toolbox package for Matlab, we simultaneously modeled twelve aspects of the task: Boundary separation (i.e., carefulness), non-decision time (i.e. RT not attributable to deciding which side is larger), starting point (i.e. bias toward choosing one kind of stimulus), drift rates for each of the six stimulus bins (i.e. the speed at which participants move toward a decision), and variability in non-decision time, starting point, and drift rate. The model held boundary separation constant across different distance bins. Given our sign conventions, positive drift rates indicate movement toward deciding the symbolic ratio is larger, and negative drift rates indicate movement toward deciding the dot ratio is larger. As with the logistic regression of the pooled data presented above, the model was based on trials aggregated across all compliant participants, and trials with RTs more than 3SD away from the mean were excluded. Results are shown in Table 4 and Fig. 4.

In previous work with whole number magnitudes, the numerical distance effect emerged in diffusion models as faster drift rates for larger distances (Ratcliff et al., 2012). We found this same pattern

Table 4
Diffusion models of performance in Experiments 1, 2, and 3 using pooled data from all compliant participants.

	Experiment 1	Experiment 2	Experiment 3
Modeled parameters			
Boundary separation	0.256	0.212	0.210
Non-decision time (s)	0.456	0.450	0.541
Variability in non-decision time	0.409	0.453	0.585
Starting point	0.113	0.102	0.106
Variability in starting point	0.122	0.098	0.141
Variability drift rate	3.04×10^{-7}	4.23×10^{-8}	0.022
Drift rates for ratio difference bins			
<−.33 (M = −.474)	−0.110	−0.133	−0.137
−.33 to −.17 (M = −.247)	−0.083	−0.097	−0.083
−.17 to 0 (M = −.101	−0.054	−0.061	−0.043
0 to .17 (M = .101)	−0.013	−0.009	0.033
.17 to .33 (M = .247)	0.018	0.048	0.097
>.33 (M = .474)	0.067	0.112	0.189
Regression predicting drift rates from mean bin value			
Slope (SE)	0.190*** (.009)	0.265*** (.018)	0.349*** (.021)
Intercept (SE)	−0.029*** (.003)	−0.023* (.006)	0.009 (.007)
R^2	0.990	0.983	0.986

* $p < .05$.
*** $p < .001$.

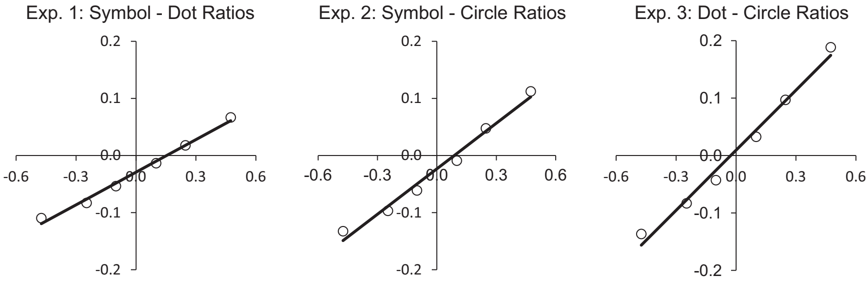

Fig. 4. Drift rates calculated from pooled data for Experiments 1, 2, and 3. The *x*-axis indicates the mean difference between the ratios for a given bin. The *y*-axis indicates drift rates. Positive drift rates indicate movement toward Symbol responses in Experiments 1 and 2, and movement toward Dot Reponses in Experiment 3. Negative drift rates indicate movement toward the Dot responses in Experiment 1, and movement toward the Circle Reponses in Experiments 2 and 3. Circles (○) indicate specific modeled drift rates. Lines indicate regressions predicting modeled drift rates from mean bin values.

when modeling our ratio data. Drift rates increased as the difference between the symbolic and dot ratios increased. We also found a significant negative *y*-intercept indicating that, indeed, the dot-ratios were perceived as larger than equivalent symbol ratios. We found similar results when we modeled drift rates for each participant separately (see Supplement S1).

2.3. Discussion

Performances on cross format comparisons were consistent with participants basing their comparisons on amodal ratio values. First, when comparing dot ratios with symbolic fractions, participants demonstrated performance well fit by sigmoid curves that typify psychophysical functions. Although participants demonstrated a bias in judging sizes across formats, this bias was sufficiently consistent that it could be accounted for by adjusting ratios values by a constant of .21. Second, although the cross-format nature of the task made analyzing numerical distance effects from raw error inappropriate, bias corrected calculations based on the PSE demonstrated classic distance effects, in both error rates and reaction time.

These results suggest that participants used perceptual pathways to access the fractional values of the nonsymbolic stimuli involved. Three facts buttress this interpretation of our results: (1) There was no simple componential route available to convert the non-symbolic stimuli to symbolic fractions. The nonsymbolic stimuli instantiated fraction stimuli using components that varied considerably from ratio to ratio, including controls for area and for correlation of components to total magnitude. (2) The components of the nonsymbolic stimuli could not be serially counted in the 1406 ms taken on average trials. Thus, there is little possibility that participants converted dot ratios to symbolic form via some count-based method. (3) Prior research using a similar paradigm with natural number values (Halberda et al., 2008) found that RTs for single-format comparisons of dot arrays were only ~300 ms shorter than they were for the current task. Given that the current comparisons were across formats and that each individual member of a stimulus pair was itself composed of two components, the additional 300 ms would seem to preclude the use of conscious algorithms for processing nonsymbolic ratios.

We conducted a supplemental experiment to gather additional support for our conclusion that participants were not converting nonsymbolic ratios to symbolic form prior to making comparisons (see Supplement S2). We had participants complete a series of different ratio comparisons so we could analyze RT transaction costs for different format combinations. We found that cross-format comparisons between symbolic and nonsymbolic forms took no longer than comparisons between pairs of symbolic fractions. Moreover, participants actually took significantly *longer* to make comparisons between pairs of symbolic fractions than they took to make comparisons between pairs of nonsymbolic ratios of the same format. These patterns are inconsistent with conversion of nonsymbolic components to symbolic form prior to comparison, because converting components should have imposed some conversion cost in terms of RT. It instead appears that participants accessed the values of nonsymbolic stimuli via some perceptual route. This perceptually based access to nonsymbolic analogs of fractions stands in contrast to arguments that there is no intuitive perceptual basis upon which to build fraction knowledge (e.g., Dehaene, 1997; Feigenson et al., 2004; Gelman & Williams, 1998)

We recognize that our method for controlling dot array areas may invite the charge that other visual aspects such as contour length and average dot size could be used to complete the task instead of the numerosities of arrays *per se*. Indeed, formulating controls for dot arrays is a vexing issue – Gebuis and Reynvoet (2012) have cogently argued that controlling for such parameters is nearly intractable, concluding "that it is unlikely that a special mechanism exists that can process nonsymbolic number independent of its visual cues" (p.647; see also De Smedt, Noël, Gilmore, & Ansari, 2013). Fortunately, this is not a critical issue for our theory: Our theory is that nonsymbolic ratio magnitudes can be perceived in multiple formats, including ratios instantiated using continuous forms. Thus, even if parameters other than numerosity are used to extract the magnitudes of dot array ratios, it does no damage to our position. Indeed, Experiment 2 was conducted with ratios made with circles for the specific purpose of showing that this ratio perception applies with continuous variables in addition to discrete numerosities.

3. Experiment 2

A multitude of prior studies from over a century of research in psychophysics have demonstrated that humans cannot accurately estimate the number of dots in arrays of the size employed in Experiment 1 (e.g., Crollen, Castronovo, & Seron, 2011; Indow & Ida, 1977; Izard & Dehaene, 2008; Kaufman, Lord, Reese, & Volkmann, 1949; Krueger, 1984; Taves, 1941). Estimates typically underestimate the actual numerosities and vary extensively both within and between subjects. Nevertheless, it might still be argued that participants in Experiment 1 estimated the numerosities of individual components of dot arrays and used these estimates as a basis for comparing symbolic and nonsymbolic ratios rather than perceiving the values of dot ratio stimuli holistically. To circumvent these arguments, in Experiment 2 we used ratios composed of circle areas as a stronger test of the ability to perceive nonsymbolic ratio magnitudes. Because circle areas – unlike dot arrays – do not correspond to any particular numbers, the argument for componential conversion to symbolic numbers for conscious use in algorithms seems less plausible. Moreover, because the numerator and denominator components for any given ratio stimulus are of different sizes, there is no way to partition circles such that count based strategies are plausible.

3.1. Method

3.1.1. Participants

Participants were a separate sample of 27 undergraduate students from a major Midwestern university, participating for course credit (25 female; ages $M = 19.6$, range = 18–22).

3.1.2. Materials and procedure

Each participant completed all comparisons in one hour-long session that also included other comparison tasks not discussed in this report. The experimental procedure was identical to that of Experiment 1 with the exception that nonsymbolic ratios were composed of circles instead of dots. Each stimulus was composed of a pair of circles separated by a horizontal bar in the middle to form a fraction, with a 2% tolerance (Fig. 5). The ratios instantiated were in terms of circle areas corresponding to the same values as the dot array stimuli of Experiment 1 (see Table 1). Circles were black against a white background. Component denominators varied among stimuli within each set. Circle diameters ranged from 7.6 mm to 42.0 mm. As in Experiment 1, each nonsymbolic ratio stimulus occupied a 54×109 mm space.

As with dot ratios, a random number generator was used to create two separate sets of stimuli: one set such that the areas of the numerator components were uncorrelated with overall ratio magnitude and a second set such that both denominator areas and total summed component areas were uncorrelated with overall magnitude. When circle ratios were presented on the left side of a comparison,

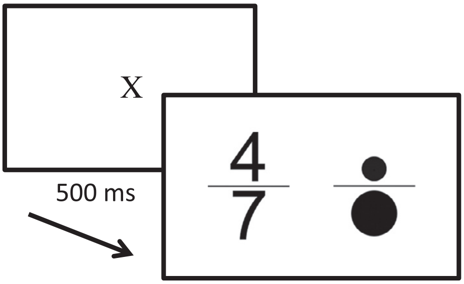

Fig. 5. Experiment 2: Sample depiction of a symbolic fraction vs. circle ratio comparison trial. Participants answered by pressing a button box indicating which stimulus they thought had the larger fractional value.

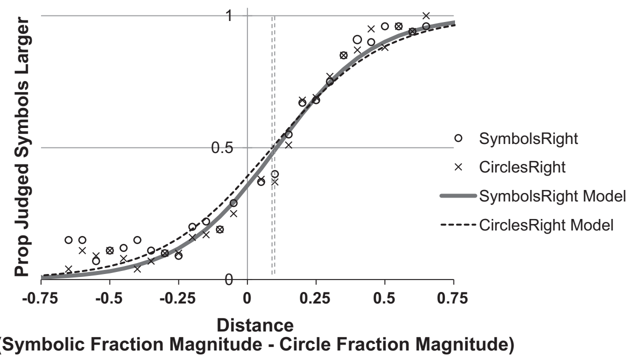

Fig. 6. Experiment 2: Proportion of trials in which participants judged the symbolic fraction to be larger than the nonsymbolic circle fraction as a function of the difference between stimuli magnitudes and whether the symbolic fraction or the circle fraction was presented on the right Each plotted point represents the mean proportion of times respondents selected the symbolic fraction as greater. The solid and dashed curves represent the estimated logistic fits to the data when symbolic fractions or circle fractions are presented on the right respectively. The dotted vertical lines mark the PSEs of .09 when the symbols were on the right and .11 when the circles were on the right. *Note:* Distances were binned to the nearest .05 for illustrative purposes in the figure but were entered exactly into the regressions.

stimuli were selected from the set with uncorrelated denominator and summed. When dot ratios appeared on the right, stimuli were sampled from the set with uncorrelated numerator areas.

3.2. Results

3.2.1. The psychophysical function

Analyses paralleled those of Experiment 1. Prior to analyses, responses with reaction times that were more than 3 SDs beyond the mean were removed. This resulted in a loss 1.4% of data points (70 of 4860). We used logistic regression using all trials as data points with likelihood of judging the symbolic fraction stimulus to be larger as the dependent variable. The independent variables were: (1) *Distance*, calculated by subtracting the magnitude of the dot stimulus in each comparison pair from the magnitude of the symbolic stimulus, (2) whether symbolic fractions or nonsymbolic circle fractions were presented on the right (SymbolsRight = 1), and (3) a *Distance* × *SymbolsRight* interaction term. Although some participants showed poor model fit, there were no instances of instruction reversal such as those seen in Experiment 1, so no additional data were trimmed.

The logistic function explained much of the variance in responses, with Nagelkerke's Pseudo $R^2 = .46$. As with dot ratio stimuli from Experiment 1, participant judgments exhibited a clear and consistent bias. On average, participants' subjective perceptions were such that circle ratios were

Table 5
Experiment 2, symbols vs. circles: results of logistic regression of probability of judging symbolic stimulus larger against distance and side of presentation – pooled and person-level data.

Parameter	Estimate
Pooled data: 4790 data points	
Nagelkerke Pseudo R^2	0.46
Constant (*SE*)	−0.60 (.06)**
Distance (*SE*)	5.65 (.23)**
SymbolsRight (*SE*)	0.15 (.08)*
Distance × SymbolsRight (*SE*)	−0.66 (.31)*
$PSE_{SymbolsRight}$	.09
$PSE_{CirclesRight}$	.11
Person level data: 180 data points	
Median Nagelkerke Pseudo R^2 (*SD*)	.59 (.22)
% of sample with significant effect for Distance	92.6% (25/27)
% of sample with significant effect for SymbolsRight	11.1% (3/27)
% of sample with significant effect for interaction	3.7% (1/27)
Mean PSE SymbolsRight (*SE*)	.08 (.03)
Mean PSE CirclesRight (*SE*)	.08 (.04)

* $p < .05$.
** $p < .01$.

judged to be larger than symbolic stimuli of the same absolute magnitude. The perceived size of a given circle ratio stimulus relative to the corresponding symbolic stimulus was its actual magnitude plus approximately .09 when symbols were presented on the right and .11 when instead circles were presented on the right. There was also a small but significant interaction between distance and the side on which circles were displayed ($p = .03$). The function was slightly steeper when participants saw symbolic fractions on the left compared to when they saw circle ratios on the left. However, this difference was quite small in substantive terms, so the functions remained nearly identical (see Fig. 6). Thus, across the range of stimuli, bias was estimated as approximately .10. Effects from summary level analyses were mirrored by analyses at the individual level. Logistic functions fit separately for each individual participant yielded patterns similar to the aggregate for the vast majority of individuals (see Table 5).

3.2.2. Adjustments for bias

Analysis of distance effects mirrored that for dot ratios. First we applied adjustments to circle ratio magnitudes, adding .10 to each to yield *perceived magnitude*, whereas symbolic stimuli received no such adjustments. Next, we calculated *adjusted distance* and *adjusted error* based on the new perceived magnitudes and their corresponding adjusted distances (defined as symbolic magnitude – perceived circle ratio magnitude). The resulting adjusted error patterns were symmetrical about the adjusted distance of zero as shown in Fig. 7. As in Experiment 1, we performed supplemental checks using reaction times as a face validity check on our bias analyses, and this analysis yielded similar results as shown in Fig. 7b and c.

3.2.3. Distance effects

We conducted group and individual level analyses mirroring those of Experiment 1. We regressed adjusted error rates and reaction times against: (1) *absolute adjusted distance*, calculated by taking the absolute value of adjusted distance as described in Section 3.2.2, (2) *SymbolsRight*, as specified in Section 3.2.2, and (3) a *Distance* × *SymbolsRight* interaction term. We also ran parallel regressions using the logarithm of adjusted interstimulus distance as the independent variable in both error and RT regressions. Trials on which participants answered incorrectly (after adjustments) were excluded from the analysis of reaction times.

3.2.3.1. Adjusted error rate analysis. The mean adjusted error rate was .21 ($SD = .41$). Group level analysis revealed that error rates fell as absolute distance between stimuli increased ($\beta_{lin} = -.79$, $p < .01$;

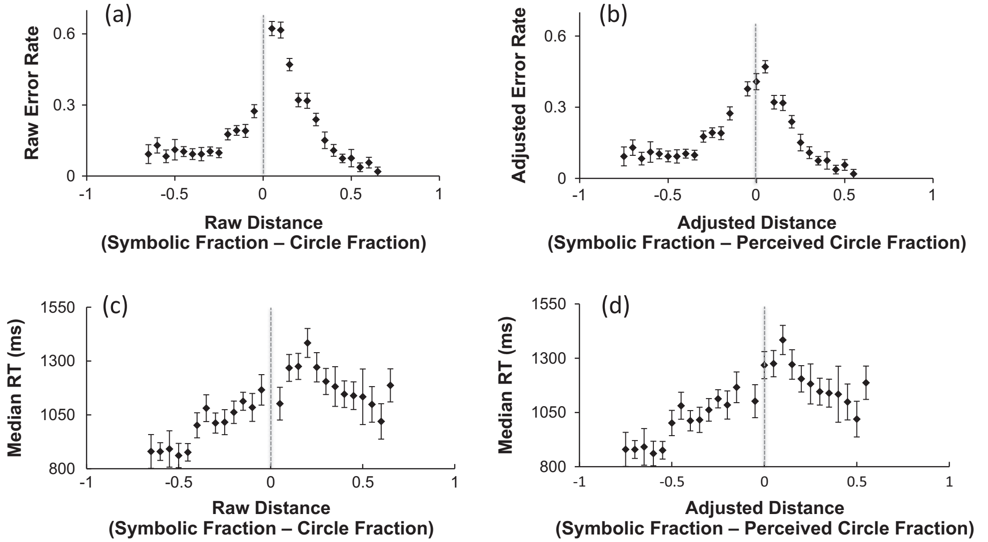

Fig. 7. Experiment 2: Error rate and RT patterns. (a) The graph of raw error rate displays a discontinuous jump due to the cross-format bias, with artificially high errors when symbolic stimuli are larger and artificially low errors when circle stimuli are larger. (b) Adjusted errors, calculated on the bias-corrected adjusted distance scale, are symmetrical about zero and reach a maximum of .5, regardless of the format of the larger stimulus. (c) Symmetry of the RT prior to bias correction is centered about approximately .1 to .2, confirming the bias of perceived equality. (d) By contrast, performance is centered closer to zero on the bias corrected distance scale. Error bars reflect standard errors of the means. *Note:* Distances were binned to the nearest .05 for illustrative purposes in the figure but were entered exactly into the regressions.

Table 6
Experiment 2, symbols vs. circles: results of linear regression of error and response time against distance and side of presentation – pooled and person-level analyses.

Level of analysis, independent variables, and coefficients	Dependent variables			
	Linear regression		Logarithmic regression	
	Mean errors	Median RT	Mean errors	Median RT
Group level				
Adj. R^2	.53	.38	.66	.27
β Distance	−.79**	−.58**	−.89**	−.53**
β SymbolsRight	−.06	.03	.15	−.06
β Distance × SymbolsRight	.12	−.10	.16	−.02
Person level				
Median adj. R^2	.10	.07	.11	.04
SD of adj. R^2	.05	.07	.07	.07
% of sample with significant effect for Distance	88.9 (24 of 27)	51.9 (14 of 27)	81.5 (22 of 27)	37.0 (10 of 27)
% of sample with significant effect for SymbolsRight	3.7 (1 of 27)	7.4 (2 of 27)	7.4 (2 of 27)	0 (0 of 27)
% of sample with significant effect for interaction	3.7 (1 of 27)	3.7 (1 of 27)	7.4 (2 of 27)	0 (0 of 27)

Note. All coefficients are standardized.
** $p < .01$.

$\beta_{log} = -.89$, $p < .01$). Adjusted errors decreased from highs around 50% for comparisons with adjusted distances near zero to rates below 10% for comparisons over larger distances. There was neither a main effect for whether circles ratios were presented on the right or left ($\beta_{lin} = -.06$, $p = .65$; $\beta_{log} = .15$, $p = .16$) nor an interaction between distance and presentation side ($\beta_{lin} = .12$, $p = .40$; $\beta_{log} = .16$, $p = .19$).

All told, there was a pronounced effect of distance on accuracy, and the order of stimulus presentation did not moderate the effect. Individual level analyses paralleled these results (Table 6).

3.2.3.2. Reaction time analysis. Patterns with RT were consistent with those found for error rates (Table 6). Moreover, the median RT was 1097 ms ($SD = 910$), nearly identical to the ∼1100 ms participants took when comparing individual dot arrays in Halberda et al. (2008), a single-format comparison task. As with Experiment 1, this would seem to preclude the use of conscious algorithms for processing nonsymbolic ratios. Thus, as with Halberda et al.'s results, our findings indicate participants made comparisons based on analog judgments.

3.2.3.3. Diffusion analysis. We again used a diffusion model to simultaneously analyze RT and accuracy data, using the same process as in Experiment 1, except here using data from comparisons between symbolic and circle ratio stimuli. Again, drift rates increased as the difference between the symbolic and nonsymbolic circle ratios increased. There was also a significant negative *y*-intercept indicating that, indeed, the circle ratios were perceived as larger than equivalent symbol ratios (see Table 4 and Fig. 4). Finally, similar results were found modeling drift rates for each participant separately (see Supplement S1).

3.3. Discussion

The findings of Experiment 2 extend those of Experiment 1 to a new class of nonsymbolic fractions – those composed of circle areas. When making cross-format comparisons, participants exhibited patterns of behavior consistent with their having the ability to perceive holistic fraction values when presented nonsymbolically. First, when comparing circle ratios with symbolic fractions, participants demonstrated performance well fit by sigmoid curves that typify psychophysical functions. Although participants demonstrated a bias in judging sizes across formats, this bias was sufficiently consistent that it could be corrected for by adjusting circle ratio values by a constant of .10. Accordingly, the adapted protocol for analyzing distance effects yielded results consistent with participants accessing the holistic magnitudes of the nonsymbolic ratio stimuli.

Second, because nonsymbolic circle stimuli could not easily be converted to symbolic form via enumeration or symbolic calculation strategies, these results suggest that participants used a perceptual pathway to access the fractional values of the stimuli presented. This has an important additional implication: The continuous nature of the stimuli is an important consideration, because ratios made of continuous stimuli can extend beyond rational numbers to include all positive real number values.

Finally, these cross format comparisons took no longer than comparisons of pairs of symbolic fractions (Supplement S2). Hence, they were completed in a time course that would seem to preclude use of conscious algorithmic strategies for converting nonsymbolic ratios into symbolic form. Thus, we interpret our findings to indicate that participants made comparisons based on analog judgments when judging circle ratios just as with dot ratios.

4. Experiment 3

For comparisons in Experiments 1 and 2, at least one stimulus from each pair was a symbolic fraction. Thus, arguments positing that participants had perceptual access to analog fractional values are somewhat muddled; only half of the stimuli in each task demanded that participants access perceptual pathways to the fraction magnitudes involved. The potential remains that there may have been some unknown pathway by which participants could leverage symbolic knowledge of Arabic fraction stimuli to approximate the values of nonsymbolic fractions. Experiment 3 therefore explored comparisons of fraction values across two different nonsymbolic formats. This manipulation required perceptual processing to access the nonsymbolically instantiated fraction values for both stimuli in a comparison set. If nonsymbolic magnitudes are accurately mapped across dissimilar formats, it would provide strong evidence that participants are accessing an abstract fractional magnitude – one that is indexed by multiple nonsymbolic instantiations.

4.1. Method

4.1.1. Participants

Participants were a separate sample of 51 undergraduates students from a major Midwestern university, participating for course credit (44 female; ages *M* = 19.8, range = 18–24).

4.1.2. Materials and procedure

Each participant completed all comparisons in one hour-long session that also included other comparison tasks not discussed in this report. The circle and dot ratio stimuli used were the same as those used in Experiments 1 and 2. The experimental procedure was identical to those of Experiments 1 and 2, with two exceptions. First, comparisons were always made on nonsymbolic pairs consisting of one circle ratio and one dot ratio (Fig. 8). Second, because two sets of test stimuli were created both for circle and for dot ratios, we took additional steps to balance the controls used. Dot ratios with uncorrelated numerators were always compared with circle ratios with uncorrelated denominators. Similarly, dot ratios with uncorrelated denominators were always compared with circle ratios with uncorrelated numerators. All 90 possible magnitude combinations were presented two times each – once with dots on the left and once with dots on the right – for each of these two types of control pairings. This resulted in a total of 360 trials per participant. Trial order was randomized, and each participant saw all the same stimuli. In parallel with Experiments 1 and 2, participants were instructed to "try to feel out the ratio instead of counting or trying to apply a formula." Trials were randomized, and each participant saw all the same stimuli.

4.2. Results

4.2.1. The psychophysical function

Analyses paralleled those of Experiments 1 and 2. Prior to analysis, responses with reaction times that were more than 3 SDs beyond the mean were removed resulting in a loss of 1.9% of all data points (357 of 18,360). We used logistic regression with likelihood of judging the dot stimulus larger as the dependent variable. The independent variables were: (1) *Distance*, calculated by subtracting the magnitude of the circle stimulus from that of the dot stimulus, (2) whether the stimulus on the right was a circle ratio or a dot ratio, and (3) a *Distance* × *CirclesRight* interaction term.

Preliminary inspection revealed two anomalies with participant performance that could compromise the validity of the results. One participant chose the dot ratio stimulus on 99.4% of all trials, regardless of the discrepancy in magnitudes. An additional 10 participants appeared to have reversed the directions, consistently choosing the smaller stimulus for at least one entire experimental block. This behavior was observed even for extreme distances (e.g. nonsymbolic ratios corresponding to 2/9 vs. 7/8), indicating that it was not due to perceptual limitations, but instead reflected a systematic reversal of response choices. We excluded these 11 non-compliant participants and ran the analysis on

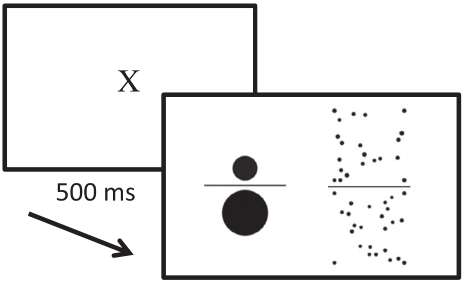

Fig. 8. Experiment 3: Sample depiction of a symbolic circle ratio vs. dot array ratio comparison trial. Participants answered by pressing a button box indicating which stimulus they thought had the larger fractional value.

Table 7
Experiment 3, dots vs. circles: results of logistic regression of probability of judging dot stimulus larger against distance and side of presentation – pooled and person-level data.

Parameter	Estimate
Pooled data	
Nagelkerke Pseudo R^2	0.50
Constant (*SE*)	.14 (.03)**
Distance (*SE*)	5.87 (.13)**
CirclesRight (*SE*)	0.00 (.04)
Distance × CirclesRight (*SE*)	−0.11 (.19)
PSE	−.02
Person level data	
Median Nagelkerke Pseudo R^2 (*SD*)	.61 (.21)
% of sample with significant effect for Distance	97.5% (39/40)
% of sample with significant effect for CirclesRight	12.5% (5/40)
% of sample with significant effect for interaction	10.0% (4/40)
Mean PSE CirclesRight (*SE*)[a]	−.03 (.02)
Mean PSE DotsRight (*SE*)[a]	−.03 (.02)

** $p < .01$.
[a] Means of individual PSEs were calculated after excluding the sole participant who failed to demonstrate a significant effect for distance. This participant had a CirclesRight PSE of −1.16, and a DotsRight PSE of −12.14.

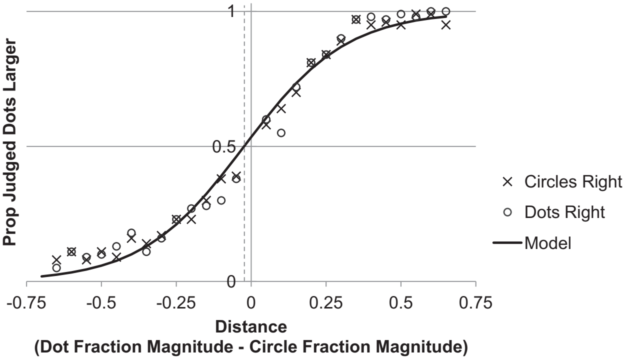

Fig. 9. Experiment 3: Proportion of trials in which participants judged the dot fraction to be larger than the circle fraction as a function of the difference between stimuli magnitudes and whether the circle fraction or the dot fraction was presented on the right. Each plotted point represents the mean proportion of times respondents selected the dot fraction as greater. The solid line curve represents estimated logistic fit to the data. Because there were no significant main effects or interactions based on presentation order, the model only predicts a single fit. The dotted vertical line marks the PSE of −.02. *Note:* Distances were binned to the nearest .05 for illustrative purposes in the figure but were entered exactly into the regressions.

the remaining analytic sample of 40 participants. Results are shown in Table 7. We only report results for the reduced sample of 40, but supplementary analyses that included reverse-coded responses for the 10 participants who reversed the directions yielded similar results.

The logistic function explained much of the variance in the responses of the analytic sample, with Nagelkerke's Pseudo $R^2 = .50$. As shown in Fig. 9, the collective sample showed a PSE of −.02 regardless of whether circles were presented on the left or on the right. There was no interaction between distance and presentation side, so the slopes of the functions were statistically identical. This means each was essentially a translation of the other by a constant factor of about .02. Logistic functions fit separately for each individual participant yielded patterns similar to the aggregate analysis for the overwhelming majority of individuals (Table 7).

4.2.2. Adjustments for bias

Given the very small absolute size of the bias, we decided that there was no need to apply bias corrections before calculating distance effects. Indeed, Fig. 10 shows that the raw data patterns both for RT and for error appear to be symmetrical about a distance of zero, serving as a face validity check that there was no need to include bias corrections. We were therefore able to examine distance effects using the traditional methods with unadjusted error rates and distances.

4.2.3. Distance effects

We conducted group and individual level analyses parallel to those of Experiments 1 and 2 (Table 8). We regressed error rates and reaction times against: (1) *Absolute distance*, calculated by taking the absolute value of distance as specified in Section 4.2.1, (2) *CirclesRight*, as specified in Section 4.2.1, and (3) a *Distance* × *CirclesRight* interaction term. We also ran parallel regressions using the logarithm of interstimulus distance in place of absolute distance in both error and RT regressions. Trials on which participants answered incorrectly after adjustments were excluded from the analysis of reaction times.

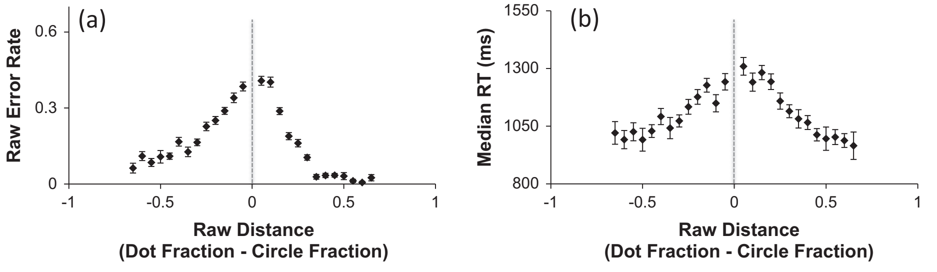

Fig. 10. Experiment 3: Error rate and RT patterns. (a) Error rates as a function of the unadjusted difference between the nonsymbolic dot and circle fraction stimuli. (b) RTs as a function of the unadjusted distance. Error and RT patterns were continuous and centered around zero on the raw distance scale, indicating a lack of any substantive bias. Error bars reflect standard errors. *Note:* Distances were binned to the nearest .05 for illustrative purposes in the figure but were entered exactly into the regressions.

Table 8
Experiment 3, dots vs. circles: results of linear regression of error and response time against distance and side of presentation – pooled and person-level analyses.

Level of analysis, independent variables, and coefficients	Dependent variables			
	Linear regression		Logarithmic regression	
	Mean errors	Median RT	Mean errors	Median RT
Group level				
Adj. R^2	.77	.56	.88	.46
β Distance	−.89**	−.79**	−.95**	−.75**
β CirclesRight	.00	.02	.04	.21
β Distance × CirclesRight	.02	.08	.03	.15
Person level				
Median adj. R^2 (*SD*)	.10 (.04)	.03 (.05)	.11 (.05)	.02 (.05)
% of sample with significant effect for Distance	90.0 (36 of 40)	62.5 (25 of 40)	92.5 (37 of 40)	62.5 (25 of 40)
% of sample with significant effect for CirclesRight	10.0 (4 of 40)	5.0 (2 of 40)	0 (0 of 40)	7.5 (3 of 40)
% of sample with significant interaction	5 (2 of 40)	2.5 (1 of 40)	7.5 (3 of 40)	2.5 (1 of 40)

Note. All coefficients are standardized.
** $p < .01$.

4.2.3.1. Error rate analysis. The mean error rate across all trials was 29.7% (SD = 45.7%). Group level analyses revealed that mean error rates fell as the absolute distance between stimuli increased (β_{lin} = −.89, $p < .01$; β_{log} = −.95, $p < .01$). Errors decreased from highs in excess of 40% for comparisons with adjusted distances near zero to rates near 5% for comparisons at the largest distances. There was neither a main effect for whether circle ratios were presented on the right or left (β_{lin} = .00, p = .99; β_{log} = .04, p = .70) nor an interaction between distance and presentation side (β_{lin} = .02, p = .88; β_{log} = .03, p = .81). All told, there was a pronounced effect of distance on accuracy, and this sole variable explained the vast majority of the variance in mean performance at the group level.

4.2.3.2. Reaction time analysis. Patterns for RT were consistent with those found for error rates (Table 8). Additionally, median RT was just under 1116 ms (SD = 846 ms). This was nearly identical to the 1100 ms participants took to compare individual dot arrays in Halberda et al. (2008), despite the fact that both comparison stimuli were composed of two separate components. Moreover, in our supplementary experiments, we found that this type of cross-format comparison was completed faster than comparisons between pairs of symbolic fractions. We interpret these findings indicate that the nonsymbolic comparisons were made using analog judgments without recourse to conscious algorithms.

4.2.3.3. Diffusion analysis. We used a diffusion model to simultaneously analyze RT and accuracy data, as in Experiments 1 and 2. Again, drift rates increased as the difference between the symbolic and circle ratios increased, as would be expected if people judge ratios based on analog magnitude (see Table 4 and Fig. 4). We found similar results modeling drift rates for each participant separately (see Supplement S1). There was no significant y-intercept, confirming that equal Dot and Circle ratios of the same value were perceived as having different equal magnitudes.

4.3. Discussion

Participants exhibited distance effects both for error rates and for RT when judging nonsymbolic ratio magnitudes across dissimilar formats. To be clear, the magnitudes compared were defined relationally by the relative sizes of components of nonsymbolic stimuli, and they could not plausibly have been deduced by conscious computational methods in the time taken. This provides strong evidence that participants had perceptual access to nonsymbolic ratio magnitudes and that these perceptually accessed magnitudes bear the same signature of analog magnitude representation seen in other perceptually oriented tasks (Moyer & Landauer, 1967; Moyer et al., 1978).

It is noteworthy that participants completed this task just as quickly as they completed the Dot vs. Arabic and Circle vs. Arabic comparison tasks, despite the fact that it involved two nonsymbolic ratios. If participants made comparisons by converting visuospatial stimuli to symbolic numbers prior to conversion, then the expectation would be that the tasks of Experiments 1 and 2 would be completed more quickly because only one of two stimuli required conversion. The speed of comparison times for nonsymbolic comparisons in Experiment 3 is consistent with conversion to the same analog magnitude code without initial conversion to symbolic numbers. Here we also underscore our finding that participants took significantly *longer* to make comparisons between pairs of symbolic fractions than they took to make comparisons between pairs of nonsymbolic ratios of the same format (see Supplement S2). Together, these results strongly suggest that participants do not first convert nonsymbolic fractions to symbolic form when making comparisons between nonsymbolic stimuli.

Finally, we found it interesting that cross-format nonsymbolic comparisons did not exhibit bias. This stands in contrast to the symbolic-to-nonsymbolic comparisons. We put forth two speculative explanations for these results. First, it may be that the two nonsymbolic formats are internally represented using the same analog magnitude code, whereas comparison of symbolic to nonsymbolic magnitudes requires a mapping step to translate symbolic fractions to the analog code. Izard and Dehaene (2008) advanced such a model for mapping between nonsymbolic numerosity arrays and symbolic whole numbers, and a similar process may be at play here. Alternatively, it may be that that comparing the two nonsymbolic formats may indeed require mapping, but that the mapping between nonsymbolic forms is more efficient than mapping between symbolic and nonsymbolic formats, leaving less space for the introduction of bias.

5. General discussion

5.1. Summary

This series of experiments presented evidence that human cognitive architectures can provide perceptual access to approximate abstract fraction magnitudes instantiated by nonsymbolic stimuli. Participants demonstrated the ability to systematically map nonsymbolic fraction magnitudes to symbolic fractions, both using stimuli composed of discrete numerosities (Experiment 1), and using stimuli composed of continuous circle areas (Experiment 2). Moreover, participants mapped between nonsymbolic ratio formats with a high degree of fidelity (Experiment 3). Response patterns for all three types of cross-format comparisons consistently bore the signature of intuitive analog magnitude representation, whether considering error rate or reaction latency. Our conclusions, accordingly, parallel those of Moyer and Landauer (1967) upon finding distance effects among Arabic numerals: "These results strongly suggest that the process used in judgments of differences in magnitude between [ratio values] is the same as, or analogous to, the process involved in judgments of inequality for physical continua" (p. 1520). These results pose a challenge to accounts that argue that human cortical structures are ill-suited for processing fractions.

This is the first line of research demonstrating such distance effects for fractional magnitudes across different notational formats (see also Matthews et al., 2014). As we noted in the literature review, the cross-format nature of the comparison is important because successful comparison within a particular format might be accomplished by methods that do not necessarily require magnitude abstraction, such as scaling. By contrast, cross-format comparisons require some sort of abstraction of magnitudes to allow comparison on the same scale.

Several features of the protocol made it unlikely that participants were able to use procedural manipulations or calculations to generate their responses. The discrete dot stimuli of Experiment 1 were too numerous and presented too briefly to be serially counted in the time participants took to complete the tasks (median RT of 1406 ms). The continuous circle stimuli used in Experiment 2 had no unique mapping to numbers, and those tasks were completed relatively briefly as well (median RT of 1097 ms). Further, participants had no tools with which to measure the diameters of the circles. The tasks of Experiment 3, which compared nonsymbolic dot and circle fractions, were also completed briefly (median RT of 1116 ms). Finally, nonsymbolic stimuli were constructed such that individual component sizes did not correlate with overall fraction magnitude, eliminating auxiliary cues to fraction magnitude. All told, these features strongly militate against the possibility that participants could follow procedural routes to find the fractional values of the stimuli. Together, these performances provide evidence that humans have an intuitive sense of nonsymbolic ratio magnitude that allows them to perceive and judge fractional number values in ways similar to how the ANS allows them to perceive and judge natural number magnitudes.

5.2. The nature of cross-format comparisons

Perhaps our most notable finding is the regularity of the patterns with which participants compared magnitudes across formats. Although it is true that participants demonstrated biases for comparisons involving nonsymbolic stimuli vs. Arabic fractions, these biases were sufficiently uniform that they could be corrected by applying a constant adjustment throughout the range of stimuli used in each study. Performance clearly followed the symmetrical s-shaped psychometric sigmoid for all three types of comparisons. In fact, comparisons across nonsymbolic formats demonstrated remarkably minimal bias, indicating that mapping across these nonsymbolic formats was highly accurate. This remained true despite the use of multiple controls to ensure that component sizes would not be good cues to overall fraction magnitude. Thus, it seems that participants could access and translate among intuitive analog representations of fractional magnitudes.

Cross format comparisons also demonstrated some differences, most notably those between Experiments 1 and 2. Dot ratio vs. symbolic fraction comparisons yielded slower RTs and larger biases than the circle ratio vs. symbolic fraction comparisons. As for the RT differences, some research suggests that humans may have superior acuity for discriminating between regular areas (e.g., circles, rectangles, and ovals) when compared to acuity for numerosities, as indicated by lower weber fractions (e.g., Morgan, 2005; Nachmias, 2008; Solomon, Morgan, & Chubb, 2011). This implies that internal representations of circles may simply be less noisy, leading to superior RT. However, this explanation raises the question of why overall error rates for the two types of comparisons remained quite similar (23% vs. 21%). Alternatively, it may be that participants spent longer on dot ratio comparisons because they felt that they should be able to be more successful with dot ratios than circle ratios. After all, dot arrays could in principle be counted, even though participants did not have enough time to perform the counts. This realization may have made participants a bit more uncomfortable with the level of uncertainty involved with the task. In contrast, participants may have been more willing to yield that circle ratio estimates were uncertain.

As for the differences in bias between Experiments 1 and 2, we once again appeal to the model put forth by Izard and Dehaene (2008). In this model, internal magnitudes are represented on an internal continuum or number line. Converting from this nonsymbolic continuum to number symbols involves mapping from this internal representation to a grid composed of symbolic numbers. In the absence of calibration, this symbolic grid is assumed to be some idiosyncratic affine transformation of the canonical internal continuum (i.e. it is some linear transformation of the internal continuum). Thus, the translation from nonsymbolic to symbolic form is generally expected to exhibit some sort of bias.

We speculate that for a given class of nonsymbolic stimuli, the affine transformation requires selection of a modulus from which all other values are scaled. We interpret the data as suggesting that in this case participants tend to select these moduli such that the bias is higher for dot ratios than for circle ratios. We previously found data consistent with this pattern using similar stimuli in numerical estimation tasks (Chesney & Matthews, 2012). When we asked participants to give numerical estimates for dot array ratios and for circle array ratios, slopes for both sets of estimates were near one, but dot ratios had a higher y-intercept. At this time, we have no well developed theory regarding why the modulus for dot ratios should necessarily be higher than that for circle ratios.

Our speculations provide directions for future investigation, but the precise nature of the perceptual processes involved remains unclear. That said, our design does allow us to draw some conclusions about the likelihood that participants proceeded by first converting nonsymbolic ratios to their symbolic equivalents. Two aspects of the results are inconsistent with such an account. First, comparisons of dot ratios vs. circle ratios – two nonsymbolic formats – took no longer than comparisons of dot ratios to symbolic fractions or circle ratios to symbolic fractions. If both nonsymbolic terms were sequentially converted to symbolic equivalents, the expectation would be that comparisons requiring conversion of two nonsymbolic stimuli would take longer than the comparisons requiring converting only a single nonsymbolic stimulus.

Second, our supplemental experiment using both within format and between format comparisons found that comparisons between two symbolic fractions took significantly longer to complete than comparisons between nonsymbolic ratios (see Supplement S2). Even cross-format comparisons between dot and circle ratios were completed significantly faster than comparisons between symbolic fractions. This stands as strong evidence that participants were not converting nonsymbolic stimuli into symbolic form in order to perform comparisons.

5.3. A generalized number less than one?

These findings also raise questions concerning Kallai and Tzelgov's (2009) claim that analog representations of fractions are generalized as a numerical value less than one. Kallai and Tzelgov investigated whether comparisons in which participants were asked to choose the larger of two symbolic fractions would be influenced by the congruence of the overall physical sizes in which fractions were presented and vice versa. They found that physical size congruity failed to influence comparisons of fraction pairs and that the magnitudes of symbolic fractions failed to influence a physical size comparison task. This led them to conclude that humans do not easily process the sizes of specific symbolic fraction magnitudes, and that this is largely due to the fact that fractions are composed of components (i.e., numerators and denominators) whose processing is more immediate and poses interference for processing of holistic fraction magnitudes. Our results address Kallai and Tzelgov's conclusion in at least two ways.

First, our results mark one case in which processing of holistic fraction values is quite specific and relatively rapid. To be sure, distance effects are based on distances between specific values, and it is hard to see how a generalized fraction magnitude would be compatible with the observed distance effects. It seems that comparing symbolic fractions with nonsymbolic equivalents, rather than with other symbolic fractions, may be a fertile ground for exploring how symbolic fraction magnitudes are represented.

Second, we suggest that the lack of a size congruity effect might be explained by the fact that fraction magnitudes are defined relationally by the ratio between two stimuli (the numerator and the denominator) whereas the size manipulations Kallai & Tzelgov used only involved total nonsymbolic magnitude (total summed area of both components), where the sizes of the numerators and denominators within a fraction were held constant (e.g. 4/5 vs. 3/7). An alternative manipulation that instead varied the component sizes within each fraction, instantiating the larger symbolic fraction with the smaller physical ratio (e.g., 4/5 vs. 3/7), might reveal a ratio-based size congruity effect. We currently have studies in progress with data to date supporting this hypothesis.

5.4. Open questions

This research raises many questions for future investigation regarding the nonsymbolic ratio processing system. One set of questions regard the nature of the interface between ratio perception, symbolic fraction knowledge and the pedagogical processes that might link the two. For instance, Dehaene's (1997) charge that human neural structure is incompatible with fractions was in part based on the fact that learners often find it quite difficult to gain a correct understanding of symbolic fractions. If, as our results and others suggest (e.g., Boyer, Levine, & Huttenlocher, 2008; Duffy et al., 2005; Vallentin & Nieder, 2008), people come equipped with a cognitive apparatus that processes fractional values in nonsymbolic form, why do people encounter such difficulties understanding symbolic fractions?

We suggest that these difficulties may result in part because the most common methods of teaching do not optimally engage the intuitive ratio processing system. For instance, the majority of current educational initiatives teach fractions using either a sort of equipartitioning or an equal sharing logic that taps counting skills and understanding of whole-number magnitudes (e.g., Cramer, Post, & delMas, 2002; Empson, 1999). It may be that these processes encourage counting and thereby discourage use of the perceptually based ratio processing system. Indeed, past work has shown that young children perform significantly worse on ratio matching tasks when partitioned, countable stimuli are used than when continuous stimuli are used (Boyer & Levine, 2012; Boyer et al., 2008; Jeong et al., 2007).

It remains to be shown whether we can leverage this perceptual sensitivity for ratios to promote learning about fractional symbols. It may be that the optimal way to foster such an appreciation is to use perceptual learning techniques (Goldstone, Landy, & Son, 2010; Kellman, Massey, & Son, 2010; Kellman et al., 2008). That is, it may prove effective to build symbolic fraction knowledge upon perceptual sensitivity to nonsymbolic fractional values by way of multiple and varied perceptual exemplars, akin to word and category learning (e.g., Markman & Wachtel, 1988). Thus, we might eventually come to teach what a fraction symbol like 1/3 represents in much the same way that we teach young children what the symbol 4 represents or what a 'dog' or a 'cat' is.

To our knowledge, Fazio et al. (2014) is the only journal published study to have investigated the relations between nonsymbolic ratio sensitivity and symbolic fraction knowledge (see also Lewis, Matthews, & Hubbard, 2014). Fazio et al. found that number line estimation with symbolic fractions was indeed predictive of math achievement among 5th grade students. We note that the proportion matching account put forth by Barth and Paladino (2011) would imply that this is precisely the sort of task that assesses the links between symbolic and nonsymbolic representations of fractions. Clearly, there is much space for exploration of the effect of interventions targeted to leverage nonsymbolic ratio abilities to help promote learners' understanding of symbolic fraction magnitudes.

This work also opens up a space for future inquiry regarding how the acuity of nonsymbolic ratio perception relates to development more generally. Are individual differences in the ability to discriminate between visuospatial fractions related to symbolic fraction knowledge test performance or to math achievement more generally? Which nonsymbolic formats are discriminated with the most acuity, and might such acuity predict pedagogical effectiveness? Perhaps most importantly, researchers need to investigate how cross-format distance effects develop with age and experience. We are currently in the beginning stages of a project investigating the developmental progression of nonsymbolic ratio sensitivity within and across formats. Answering this developmental question will be pivotal to evaluating claims such as those by Jacob et al. (2012), who hypothesized that ratio magnitude processing is a core or native competence. Although several studies suggest that young children may have this ability (e.g., Duffy et al., 2005; Singer-Freeman & Goswami, 2001; Sophian, 2000; Spinillo & Bryant, 1991), only one to date has shown this ability among infants (McCrink & Wynn, 2007). Filling this gap would well complement the primate data (e.g. Jacob et al., 2012) in casting ratio processing as a core competence. Finally we might ask, more broadly, if this ratio perception ability generalizes to non-visuospatial domains.

5.5. Conclusion

Together, these experiments provide evidence of flexible processing of nonsymbolic fractional magnitudes in ways similar to the way the ANS processes the magnitude of discrete numerosities. Just as with the ANS, there is early evidence that this ability is abstract: here it has been demonstrated with multiple instantiations composed of dots and of circle areas. These cross-format distance effects indicate that processing of ratio magnitudes bears the same signature that is typically found when other *perceptual* stimuli are used in comparison tasks. Similar findings with whole number comparisons have been widely interpreted as indicating that the processes involved mirror perceptual judgments (e.g., Dehaene, 1997; Hubbard, Piazza, Pinel, & Dehaene, 2005; Moyer & Landauer, 1967; Nieder, 2005). By extension, the current data support similar conclusions regarding factional values.

We temper our conclusions with the full acknowledgment that research foregrounding our abilities to perceive ratios *per se* is in its infancy. There remain many miles to go before we can draw conclusions about whether this ratio perception ability is truly on par with the ANS based sensitivity to nonsymbolic analogs for whole number. Indeed, our findings raise as many questions as they answer. What is certain, however, is that the study of ratio perception is full of possibilities. One such possibility is that fractions may in some sense be natural numbers too.

Acknowledgments

We thank Ed Hubbard, David Landy, Mark Lewis, David Purpura, and Victoria Simms for their helpful commentary on early drafts of this manuscript, Kevin Potter for advice on drift rate modeling, and Ellen Peters for her support. This research was supported in part by an internal grant from the Wisconsin Alumni Research Fund, in part by NSF Grant DRL-1420211, and in part by NSF-Grant SES-1155924. An earlier version of these experiments was first presented at the 36th Annual Conference of the Cognitive Science Society in 2014.

Appendix A. Supplementary material

Supplementary data associated with this article can be found, in the online version, at http://dx.doi.org/10.1016/j.cogpsych.2015.01.006.

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