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Individual Differences in Nonsymbolic Ratio Processing Predict Symbolic Math Performance

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Abstract

What basic capacities lay the foundation for advanced numerical cognition? Are there basic nonsymbolic abilities that support the understanding of advanced numerical concepts, such as fractions? To date, most theories have posited that previously identified core numerical systems, such as the approximate number system (ANS), are ill-suited for learning fraction concepts. However, recent research in developmental psychology and neuroscience has revealed a ratio-processing system (RPS) that is sensitive to magnitudes of nonsymbolic ratios and may be ideally suited for supporting fraction concepts. We provide evidence for this hypothesis by showing that individual differences in RPS acuity predict performance on four measures of mathematical competence, including a university entrance exam in algebra. We suggest that the nonsymbolic RPS may support symbolic fraction understanding much as the ANS supports whole-number concepts. Thus, even abstract mathematical concepts, such as fractions, may be grounded not only in higher-order logic and language, but also in basic nonsymbolic processing abilities.

Keywords

number comprehension, educational psychology, individual differences, mathematical ability, perception

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> What is a number, that a man may know it, and a man, that he may know a number?
>
> —Warren McCulloch (1960)

Developmental and cognitive psychologists have long debated McCulloch's question, with some arguing that numerical cognition is constructed in a top-down manner using higher-order cognitive abilities (Piaget, 2013; Rips, Bloomfield, & Asmuth, 2008; Wiese, 2003), whereas others have argued that numerical cognition is grounded in a primitive nonsymbolic *number sense* (Feigenson, Dehaene, & Spelke, 2004; Gallistel & Gelman, 2000; Nieder, 2005).[1] Cognitive-primitive accounts have often been deemed inadequate for explaining how humans understand numbers other than whole numbers, such as fractions or square roots (but see Barth, Starr, & Sullivan, 2009; McCrink, Spelke, Dehaene, & Pica, 2013). Some cognitive-primitive theorists argue that limitations of core numerical systems constrain the number concepts that are intuitively accessible (Feigenson et al., 2004; Wynn, 1995). For instance, Dehaene (2011) concluded that numbers such as fractions "defy intuition because they do not correspond to any preexisting category in our brain" (p. 76). Consequently, such accounts have largely given way to logical and linguistic accounts with respect to more advanced numerical concepts: Numbers such as fractions are logical constructs that can be understood only with great difficulty by recycling whole-number schemas.

Here, we present evidence that cognitive-primitive accounts can be expanded to apply to fractions and perhaps even to all positive real numbers. We demonstrate that individual differences in a hitherto underappreciated

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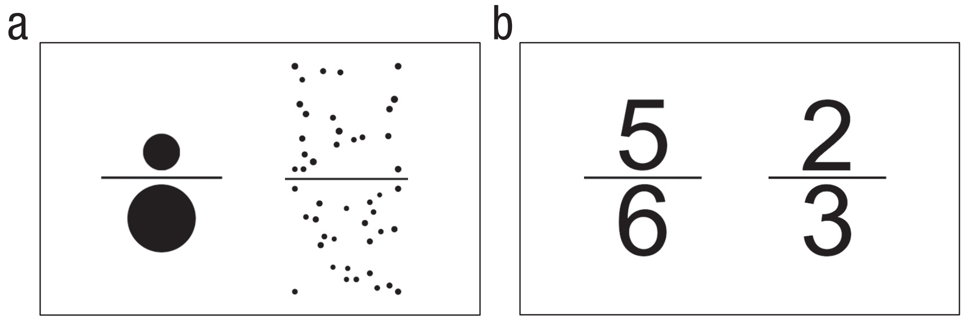

Fig. 1. Sample stimuli from tasks involving (a) nonsymbolic- and (b) symbolic-ratio comparison (taken from Matthews & Chesney, 2015). For both types of tasks, participants are asked to indicate whether the ratio expressed on the left or on the right is greater. Matthews and Chesney (2015) found that participants complete nonsymbolic-ratio comparisons more quickly than symbolic-ratio comparisons, which suggests that nonsymbolic-ratio comparisons are made without converting ratios to symbolic form.

nonsymbolic *ratio-processing system* (RPS) predict differences in higher-order math skills, including symbolic-fraction competence and algebra achievement scores.

Neurocognitive Start-Up Tools for Fractions Concepts?

Advances in cognitive and neural sciences have allowed researchers to chart what Piazza (2010) dubbed "neurocognitive start-up tools" that provide the foundations of numerical cognition. These tools include the object-tracking system (OTS), which supports rapid and exact enumeration of small sets, and the approximate number system (ANS), which enables humans and other animals to quickly approximate numerosities of large sets. It has been proposed that these basic abilities are recycled from their original functions to be used in formal learning to support acquisition of math concepts with parallel structures (Dehaene & Cohen, 2007; Piazza, 2010; but see De Smedt, Noël, Gilmore, & Ansari, 2013).

However, ANS and OTS abilities appear to provide a poor foundation for understanding numbers that are not count-based. For example, fractions such as 1/2 cannot be reached by counting. Moreover, each fractional magnitude can be represented in infinite ways (e.g., 1/2 represents the same magnitude as 2/4, 11/22, etc.). Consequently, constructing conceptions of fraction magnitude by recycling ANS- or OTS-based representations may be a difficult and error-prone process (but see DeWolf, Bassok, & Holyoak, 2015). Indeed, both children and highly educated adults often struggle to understand fractions (Lipkus, Samsa, & Rimer, 2001; Ni & Zhou, 2005). Hence, it may initially seem reasonable to conclude that these pervasive difficulties arise from innate constraints on human cognitive architecture (see Ni & Zhou, 2005, for a review).

However, there is no a priori reason to assume that these are the only intuitive building blocks available. Indeed, Jacob, Vallentin, and Nieder (2012) posited that dedicated neural networks have evolved that automatically process nonsymbolic-ratio magnitudes in multiple formats. Recent developmental and neuroscientific findings are consistent with such a potential nonsymbolic building block. McCrink and Wynn (2007) showed that 6-month-olds are sensitive to nonsymbolic ratios. By the age of 4 years, children can complete tasks requiring addition and subtraction of nonsymbolic part-whole fractions (Mix, Levine, & Huttenlocher, 1999). Moreover, research shows that both American 6- and 7-year-olds (Barth et al., 2009) and children from indigenous tribes who have limited number concepts (McCrink et al., 2013) can effectively perform multiplicative scaling with numerosities, which perhaps implicates the ANS in ratio processing. By the age of 11 years, children's performance on number-line estimation with nonsymbolic fractions is correlated with standardized test scores (Fazio, Bailey, Thompson, & Siegler, 2014). Finally, work on psychophysical scaling may also implicate nonsymbolic ratio-processing abilities (e.g., Hollands & Dyre, 2000; Stevens & Galanter, 1957).

This sensitivity to nonsymbolic ratios is flexible and abstract. Singer-Freeman and Goswami (2001) showed that children can draw proportional analogies among pizzas, chocolates, and lemonade even though the materials equated are visually dissimilar. Matthews and Chesney (2015) demonstrated that participants made cross-format nonsymbolic-ratio comparisons (Fig. 1) faster than they made symbolic-fraction comparisons, which suggests that nonsymbolic comparisons were made without first translating ratios into symbolic form. Finally, Vallentin and Nieder (2008) showed that monkeys matched nonsymbolic line pairs instantiating the same ratios even when there was substantial variation in line lengths comprising the ratios. Such flexibility across developmental time, with novel stimuli and even across species, is consistent with the existence of an RPS, which may provide an intuitive foundation for understanding fraction magnitudes.

Hypothesized RPS Support for Emerging Fraction Concepts

Recent research has shown that understanding fraction magnitudes is a key factor in promoting the acquisition of other fraction concepts and procedures (Siegler et al., 2012; Siegler & Pyke, 2013). We suggest that RPS-based processing supports learners' understanding of the overall magnitude of symbolic fractions, even when formal instruction does not explicitly attempt to leverage it. Our theory (see Lewis, Matthews, & Hubbard, 2015) makes three key predictions. First, both formal and informal learning experiences help to generate links among symbolic fractions and RPS representations of nonsymbolic-ratio magnitudes. These links help ground interpretation of fraction symbols, making them meaningful (cf. Siegler & Lortie-Forgues, 2014). This influences later learning involving fraction concepts. Second, individual differences in RPS acuity can moderate effects of learning experiences, so learners with better RPS acuity should build more precise symbolic-to-nonsymbolic links, which promotes better fraction knowledge. Third, as suggested by Jacob and Nieder's (2009) demonstration that humans process nonsymbolic ratios even when viewed passively, the RPS should exert its effects on learning even when it is not an explicit pedagogical focus. Here, we tested a key prediction of our account: that individual differences in RPS acuity would predict symbolic mathematical competence.

The Current Study

Despite evidence that humans are sensitive to nonsymbolic-ratio magnitudes, the connections between RPS acuity and symbolic mathematical ability have remained largely uninvestigated (but see Fazio et al., 2014). We used a multiple regression framework to test whether RPS acuity predicts symbolic math skills.

Method

Participants

One hundred eighty-three undergraduate students at a large American university participated in the study for course credit. Two participants did not understand the directions for the computerized tasks and were excused, which left a final sample of 181 (155 females, 26 males; mean age = 19.8 years, range = 18–22).

A power analysis revealed that to detect medium-sized effects for our three symbolic fraction outcomes (Cohen's f = .15) given four predictors (RPS acuity and three control variables) would require 84 participants. We expected effects for predicting the more distal algebra score to be smaller, but there was no preexisting literature to guide expectations for how much smaller. Hence, we set out to recruit 200 participants; 183 were recruited before the end of the academic year when data collection ended.

Materials and procedure

For our predictors, we constructed four nonsymbolic-ratio comparison tasks that paralleled those typically used to measure ANS acuity. Jacob and Nieder (2009) used similar tasks to assess the neural representation of proportions, but the tasks have never been used in connection with symbolic math outcomes. In each, the participants' task was to choose the larger of two nonsymbolic ratios (Fig. 2). We determined ratio size by dividing the magnitude of the smaller component of a ratio by that of the larger. For instance, for line ratios such as those shown in Figure 2b, the length of the white line segment was divided by the length of the black line segment. Thus, the ratio on the left side is larger than that on the right side, despite the fact that the individual components composing the ratio on the right side are larger.

There were four outcome measures: three of symbolic fractions knowledge and one of algebra from participants' university entrance examination. Algebra scores served as a distal measure, because previous research has shown fraction knowledge to be a critical predictor of algebra performance (Siegler et al., 2012). We also included other cognitive tasks as covariates. Two tasks controlled for participants' abilities to process the absolute magnitudes of the components of nonsymbolic ratios in contrast to their relative magnitudes (i.e., an ANS-based numerosity discrimination task and a line-length discrimination task). A flanker task controlled for differences in inhibitory control.

Nonsymbolic comparisons (RPS acuity and control tasks). We measured RPS acuity using four tasks in which participants had to make nonsymbolic-ratio comparisons. These tasks assessed the ability to discriminate between ratios composed of dot arrays or line segments. Each type of stimulus was presented with its components both separated and integrated (see Fig. 2). We implemented several controls to reduce reliance on the magnitudes of individual components (i.e., any strategic bias to base judgments on one specific component rather than on overall ratio values) and irrelevant surface features

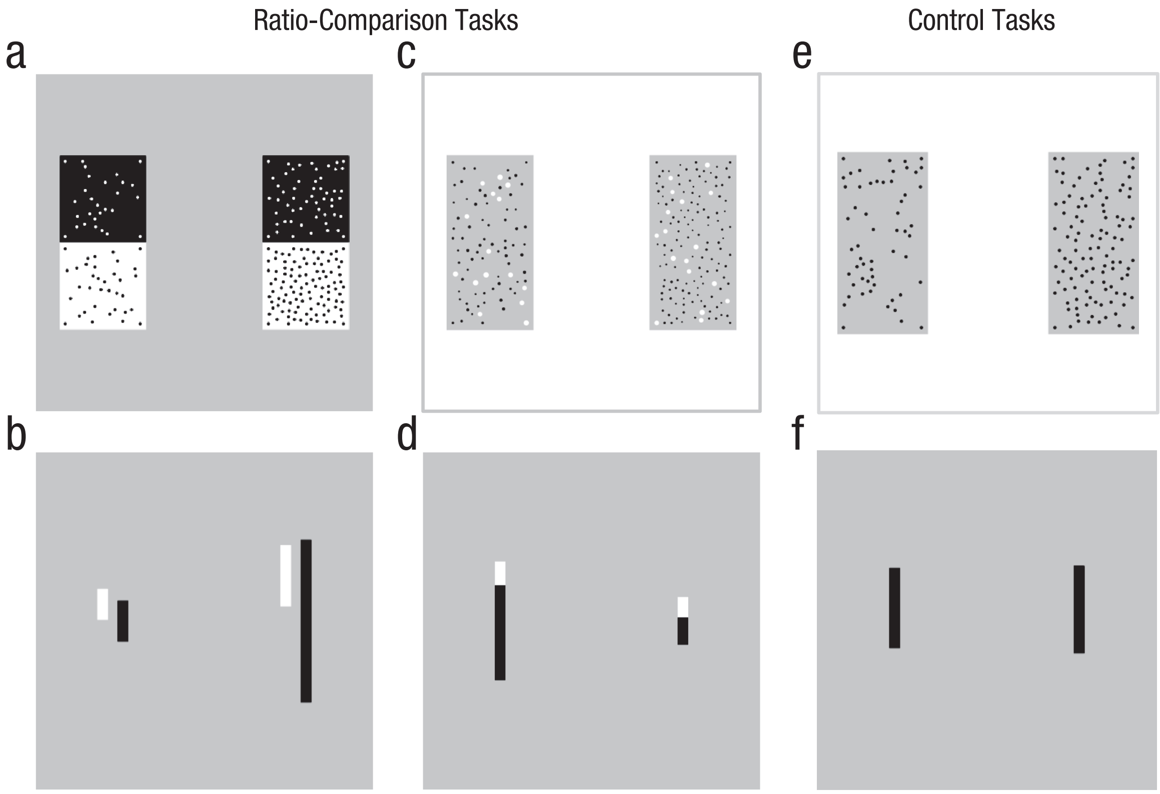

Fig. 2. Sample stimuli from the nonsymbolic comparison tasks. In the ratio-comparison tasks, participants determined whether the ratio of white dots or line lengths to black dots or line lengths was greater on the left or on the right. The white and black stimuli either appeared separately (a, b) or were integrated together (c, d). In the control tasks (e, f), only black dots and lines were presented, and participants indicated whether the number of dots or the length of the line was greater on the left or on the right.

(e.g., the sizes as opposed to the number of dots in an array). For half the trials, components of the larger ratio were shorter or less numerous than the corresponding components of the smaller ratio. In the remainder, components of the smaller ratios were shorter or less numerous than the corresponding components of the larger ratio. For dot stimuli, in half of trials, sets had the same cumulative area but variable dot sizes, such that the average dot area in an array was correlated with numerosity. In the other half, dot size was held constant, such that overall area was correlated with numerosity.

The nonsymbolic comparison tasks controlled for participants' abilities to discriminate between pairs of individual dot arrays (i.e., their ANS acuity) or pairs of individual line segments. These control comparison tasks allowed us to measure the independent contribution of RPS acuity over and above effects due to acuity for processing pairs of individual stimuli.

For nonsymbolic-ratio comparisons, participants selected the larger ratio, whereas for nonsymbolic control comparisons, participants selected the more numerous dot array or the longer line segment. For all tasks, selections were made by pressing a key ("j" to indicate that the stimulus on the right was larger and "f" to indicate that the stimulus on the left was larger). Only one type of comparison was performed per block. Each trial began with a fixation cross for 1,000 ms, immediately followed by two briefly presented comparison stimuli (Fig. 3). Because each nonsymbolic-ratio stimulus was constructed of two components (e.g., black and white line segments of differing lengths), ratio stimulus pairs were presented for 1,500 ms before disappearing; each control stimulus, in contrast, was constructed from only a single component and so was presented for only 750 ms. Trials did not progress until participants responded. There was no time limit, but a pilot study indicated that responses would be fast (< 2,000 ms on average).

Each control block consisted of 15 practice trials followed by 40 experimental trials, and each ratio-comparison block consisted of 10 practice trials followed by 45 experimental trials. Difficulty varied from trial to trial and was operationalized as the ratio between stimulus magnitudes compared in a trial, with difficulty increasing as the ratio approached 1:1. For example, a difficulty ratio of 2:1 might be instantiated by the comparison of 100 dots to 50 dots in a control comparison task. Likewise, it could be

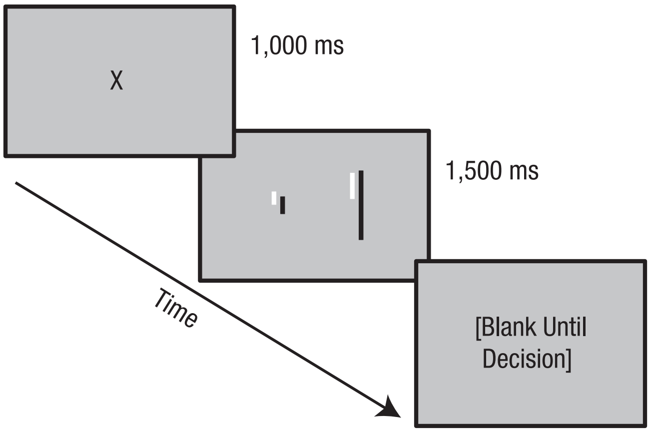

Fig. 3. Example ratio-comparison trial sequence. After viewing the stimulus array (here, two pairs of line segments with different ratios), participants indicated whether the ratio of the length of the white line segment to the length of the black line segment was greater on the left or on the right. In this case, the ratio on the left is larger, as the white line is almost the same length as the black line.

instantiated in a dot-ratio comparison task by juxtaposing 50 dots over 100 dots in one stimulus and juxtaposing 20 over 80 dots in another (i.e., nonsymbolic instantiations of 1/2 vs. 1/4). Difficulty ratios tested varied by task, because pilot experiments indicated that acuity varied from task to task. On the basis of pilot data, we chose difficulty ratios to span the spectrum from easy enough to elicit near-perfect performance to difficult enough to elicit chance performance at the group level (Table 1). All participants judged the same comparison stimuli, but the order of these stimuli was randomized for each participant.

Symbolic-fraction comparison. In the symbolic-fraction comparison task, participants selected the larger of two fractions. All fractions were composed of single-digit numerators and denominators. To reduce reliance on componential strategies (i.e., any strategic bias to base judgments on either the numerator or the denominator rather than on overall fraction values), we followed Schneider and Siegler (2010) in creating fraction pairs that shared no common components (i.e., all numerators and denominators in any given pair were unique, as in 3/4 vs. 2/5). Thirty pairs were sampled at random from all possible combinations of single-digit irreducible fractions, except 1/2, because previous work suggests that 1/2 has a special status (e.g., Spinillo & Bryant, 1991). All participants judged the same fraction pairs, but the order in which pairs appeared was randomized for each participant. Side of presentation was randomized, with the restriction that in half of the trials, the larger fraction was presented on the left.

Each trial began with a fixation cross for 750 ms, immediately followed by the fraction pair until either the participant responded or 5,000 ms had elapsed and the trial timed out. Prior to experimental trials, participants made five practice judgments.

Number-line estimation. The number-line estimation task was patterned after that of Siegler, Thompson, and Schneider (2011). Participants used a mouse to indicate the position of fraction stimuli (1/19, 1/7, 1/4, 3/8, 1/2, 4/7, 2/3, 7/9, 5/6, or 12/13) on a number line with the end points 0 and 1. Performance was defined as each participants' mean percentage of absolute error (PAE), where PAE = (|answer − correct answer|/numerical range). A smaller PAE indicates more accurate responding.

Flanker task. Prior work has shown that participants often have to inhibit their initial inclinations to respond on the basis of the sizes of fraction components rather than on the overall size of the fraction (DeWolf & Vosniadou, 2015). For example 1/4 is larger than 1/5, but participants are tempted to select 1/5 as larger, because 5 is larger than 4. Thus, it seems that individual differences in cognitive control may be important for explaining individual differences in fraction processing. We included the flanker task as a covariate to control for any such differences in cognitive control. Each trial began with the presentation of a fixation cross for 500 ms, followed by an array of five equally sized and equally spaced white arrows on a black background. Arrays remained on screen for 800 ms or until participants made a response. Participants were instructed to "choose which direction the center arrow is pointing." Participants first completed 12 practice trials and then 80 experimental trials. Forty of

Table 1. Minimum and Maximum Ratios Between Stimulus Pairs

	Dot arrays			Line segments		
Range of ratio	Control task	RPS task: integrated components	RPS task: separate components	Control task	RPS task: integrated components	RPS task: separate components
Maximum	3:1	4:1	3:1	11:10	2:1	2:1
Minimum	8:7	4:3	6:5	15:14	8:7	8:7

Note: For ratio stimuli, each cell represents a ratio of ratios (e.g., a 3/5 vs. 3/4 comparison is a 5:3 ratio). RPS = ratio-processing system.

Table 2. Bivariate Correlations Between Nonsymbolic-Ratio Comparison Measures

Task	2	3	4	5
1. Line segments: separate components	.44*	.47*	.32*	.71*
2. Line segments: integrated components	—	.46*	.43*	.79*
3. Dot arrays: separate components		—	.35*	.78*
4. Dot arrays: integrated components			—	.71*
5. RPS-acuity composite score				—

Note: RPS = ratio-processing system.
*$p < .001$.

the experimental trials were *congruent*, with the four flanking arrows pointed in the same direction as the center arrow (e.g., "> > > > >"). The other 40 trials were *incongruent*, with the four flanking arrows pointed in the opposite direction (e.g., "< < > < <"). In half of the trials, the center arrow pointed left, and in the other half, it pointed right. The order of trials was randomized for each participant. Trials in which no response was made within 800 ms were scored as incorrect. Accuracy was used as the independent variable in the analysis.

Fraction-knowledge assessment (FKA). The FKA is a 38-item pencil-and-paper assessment of both procedural and conceptual fraction knowledge. We constructed this instrument using items taken from key national and international assessments, including the National Assessment of Educational Progress and the Trends in International Mathematics and Science Study, and from instruments developed by psychology and math-education researchers (Carpenter, 1981; Hallett, Nunes, Bryant, & Thorpe, 2012). The FKA had strong internal consistency (Cronbach's α = .88).

Algebra entrance exam. The algebra entrance exam is a 35-item subtest of the university placement exam taken by all incoming freshmen. The algebra subtest has strong internal consistency (Cronbach's α = .90). The assessment was developed with scale scores normed to a mean of 500 and a standard deviation of 100. Note that analysis of the relations between RPS and algebra scores was confined to the portion of the sample who enrolled in the university system as freshmen, because exam scores were not available for 24 participants who transferred into the university after first attending other colleges.

Outlier removal and recoded cases. For all comparison tasks (symbolic and nonsymbolic), we first excluded data with response times (RTs) below 250 ms and with RTs greater than 5,000 ms (to accord with the 5,000-ms limit imposed on symbolic-fraction comparisons). Next, at the participant level, we removed trials with RTs more than 3 standard deviations faster or slower than a participant's mean RT for that task. Altogether, these steps resulted in the loss of 1.5% to 3.2% of the data for each task. Finally, for tasks other than the FKA and algebra entrance exam, we excluded data from participants who scored more than 3 standard deviations from the group mean for that task, while including those participants' data from other tasks. This resulted in the exclusion of 1 participant from the separate-component line-ratio comparisons, 1 from the ANS control task, 2 from the line-segment control comparison task, 3 from the number-line estimation task, and 5 from the flanker task.

In a few cases, participants appeared to give reversed responses. For example, 2 participants in the flanker task had less than 20% accuracy, whereas average accuracy was 92.8% (*SD* = 9.3%). These participants' responses were reverse-coded prior to analysis (2 participants for the separate-component line-ratio comparisons, 5 for the separate-component dot-ratio comparisons, and 2 for the flanker task). We also completed supplemental analyses excluding these data points instead of recoding. All significant effects for RPS acuity reported below remained significant when analyses were rerun.

Results

RPS acuity

Although the nonsymbolic-ratio comparison tasks were novel, participants effectively completed them; average accuracy for each task was 70% or higher. Moreover, accuracy was significantly correlated across different RPS tasks (Table 2). We conducted an exploratory factor analysis to assess whether these separate tasks loaded on the same latent RPS construct despite differences in format. The analysis yielded a single-factor solution (eigenvalue = 2.24). No other factors had eigenvalues greater than 1, which suggests that all four tasks measured the same factor, RPS acuity. We therefore constructed a composite RPS-acuity score, defined as accuracy averaged across all four nonsymbolic-ratio comparison tasks, for use in subsequent analyses.

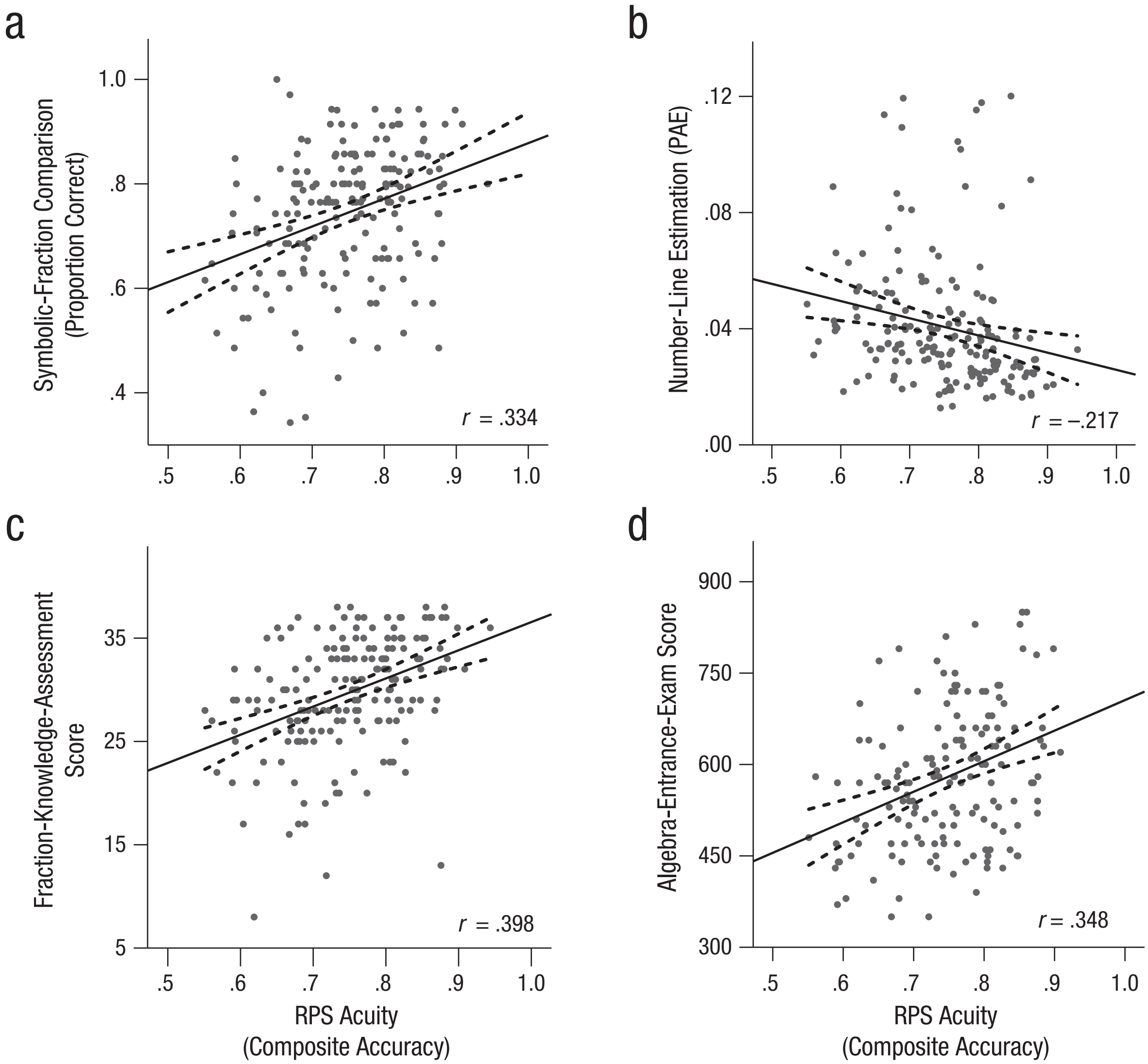

Fig. 4. Scatterplots depicting zero-order correlations between ratio-processing system (RPS) acuity and (a) symbolic-fraction comparison, (b) number-line estimation, (c) scores on the fraction-knowledge assessment, and (d) scores on the algebra entrance exam. RPS acuity was calculated by averaging accuracy across the four RPS tasks. Solid lines show best-fitting regression values, and dashed lines indicate 95% confidence intervals. In (b), a lower percentage of absolute error (PAE) denotes better performance.

Predicting symbolic math outcomes from RPS acuity

We conducted a series of two-stage hierarchical linear regressions to test whether individual differences in composite RPS acuity predicted symbolic fraction knowledge and algebra achievement above and beyond individual differences in control measures of ANS acuity, line-length acuity, and flanker-task performance (see Fig. 4 for scatterplots based on zero-order correlations). In the first stage of each analysis, we entered the control variables ANS acuity, line discrimination, and flanker-task performance to yield a base model. In the second stage, we added composite RPS acuity. Sample sizes and regression results are reported in Tables 3 through 6. We report standardized coefficients because they facilitate interpretation of effect sizes that parallel Cohen's *d*s. Note that the outlier removal in addition to the missing FKA scores for transfer students led to different numbers of data points for regressions on different outcomes, as indicated in the tables.

Stage 1: Control measures. Performance on line-segment comparisons and on flanker tasks failed to significantly predict performance for any of the four symbolic mathematics outcome measures ($ps \geq .10$). However, ANS acuity was a significant predictor of both FKA performance ($\beta = 0.18$, $p = .033$) and scores on the algebra entrance exam in the base models ($\beta = 0.17$, $p = .050$).

Table 3. Results From the Hierarchical Regression Analysis Predicting Accuracy in Symbolic-Fraction Comparisons

Step and variable	β	*p*	sr^2
Step 1 (R^2 = .03, ΔR^2 = .03)			
Line-length acuity	0.01	.906	.00
ANS acuity	0.16	.053	.02
Flanker-task performance	–0.04	.573	.00
Step 2 (R^2 = .14, ΔR^2 = .11)			
Line-length acuity	–0.09	.285	.01
ANS acuity	0.08	.305	.01
Flanker-task performance	–0.12	.111	.01
RPS acuity composite	0.38	< .001	.11

Note: n = 174. The ratio of the squared semipartial correlation coefficient (sr^2) for ratio-processing system (RPS) acuity to the summed sr^2 for all control variables was 4.57. ANS = approximate number system.

Altogether, the control measures explained between 3% and 5% of the variance for each outcome measure.

Stage 2: RPS acuity. RPS acuity was a significant predictor of each of the outcome measures (fraction comparison: β = 0.38, number-line estimation: β = −0.22, FKA: β = 0.43, algebra: β = 0.34; *p*s < .01). Introducing composite RPS acuity into each model more than doubled the variance explained for number-line estimation and more than tripled the variance explained for the other three outcomes. Moreover, when RPS acuity was introduced, ANS acuity was no longer a significant predictor for any of the four outcomes. This indicates that much of the predictive power of ANS acuity in Step 1 was actually due to shared variance between ANS and RPS acuity. Squared semipartial correlation coefficients (sr^2) indicate the unique variance explained by each of the variables in the final model over and above any shared variance among items. In Tables 3 through 6, we also show a ratio of the sr^2 for RPS acuity to the summed sr^2 of all control variables. This value indexes how large the independent unique explanatory power of RPS acuity was relative to the summed unique contributions of the other variables. For each of the outcome variables, the unique contribution of the RPS composite was 4.5 to 9.9 times larger than the summed unique contributions of the other three variables. Indeed, the effects of the control variables were small enough that the bivariate scatterplots seen in Figure 4 are near-accurate indications of the overall relations between RPS acuity and the four outcome variables in the multiple regression framework.

Discussion

These results demonstrate that individual differences in sensitivity to nonsymbolic-ratio magnitudes predict symbolic math competence. Participants with higher RPS acuity performed better on three measures of symbolic fraction knowledge and on a measure of algebra knowledge. This correlational evidence is consistent with our hypothesis that

Table 4. Results From the Hierarchical Regression Analysis Predicting Percentage of Absolute Error in Number-Line Estimation

Step and variable	β	*p*	sr^2
Step 1 (R^2 = .03, ΔR^2 = .03)			
Line-length acuity	–0.13	.105	.02
ANS acuity	–0.05	.516	.00
Flanker-task performance	0.00	.981	.00
Step 2 (R^2 = .07, ΔR^2 = .04)			
Line-length acuity	–0.08	.351	.00
ANS acuity	–0.01	.937	.00
Flanker-task performance	0.05	.565	.00
RPS acuity composite	–0.22	.009	.04

Note: n = 172. The ratio of the squared semipartial correlation coefficient (sr^2) for ratio-processing system (RPS) acuity to the summed sr^2 for all control variables was 5.72. ANS = approximate number system.

Table 5. Results From the Hierarchical Regression Analysis Predicting Scores on the Fraction-Knowledge Assessment

Step and variable	β	*p*	sr^2
Step 1 (R^2 = .05, ΔR^2 = .05)			
Line-length acuity	0.10	.229	.01
ANS acuity	0.18	.033	.03
Flanker-task performance	–0.02	.779	.00
Step 2 (R^2 = .20, ΔR^2 = .15)			
Line-length acuity	–0.01	.881	.00
ANS acuity	0.08	.293	.00
Flanker-task performance	–0.11	.151	.01
RPS acuity composite	0.43	< .001	.15

Note: n = 171. The ratio of the squared semipartial correlation coefficient (sr^2) for ratio-processing system (RPS) acuity to the summed sr^2 for all control variables was 9.90. ANS = approximate number system.

the ability to process nonsymbolic-ratio magnitudes may be an important proto-mathematical ability on which emerging mathematical competence is built. This has four potential implications for theories of human numerical cognition.

First, humans have intuitive access to nonsymbolic-ratio magnitudes that may support symbolic number knowledge. Jacob et al. (2012) previously made a similar point, yet this position remains novel. Many cognitive scientists have argued that fractions are not directly accessible to intuition, and education researchers have argued that sound conceptions of fractions must be founded on partitioning wholes into countable elements (Pitkethly & Hunting, 1996; Steffe, 2001). Hence, although visuospatial representations, such as pies and number lines, are used extensively to teach fractions, they are typically employed in contexts emphasizing counting as opposed to contexts attempting to leverage the RPS.

Why might RPS acuity explain differences in symbolic math outcomes despite this fact? Although concrete representations are typically used to elicit whole-number concepts and procedures, RPS networks continue to encode holistic-ratio magnitudes of these representations (Jacob & Nieder, 2009). Thus, educational experiences build symbolic-to-nonsymbolic links implicitly even when the RPS is not an explicit focus, which helps to improve learners' understanding of symbolic-fraction magnitudes. This mechanism is generally compatible with accounts of how perceptual learning supports mathematical cognition (e.g., Goldstone, Landy, & Son, 2010; Kellman, Massey, & Son, 2010).

This is not to suggest that the RPS alone is sufficient for building robust fraction knowledge. As noted by Kieren (1976), robust fraction knowledge requires mastery of several interpretations, including the important part-whole interpretation that relies heavily on partitioning and counting. However, the RPS may provide a more intuitively accessible route for promoting the critical-magnitude interpretation of fractions highlighted by recent research.

Table 6. Results From the Hierarchical Regression Analysis Predicting Scores on Students' Algebra Entrance Exam

Step and variable	β	*p*	sr^2
Step 1 (R^2 = .04, ΔR^2 = .04)			
Line-length acuity	–0.03	.774	.00
ANS acuity	0.17	.050	.03
Flanker-task performance	0.08	.333	.01
Step 2 (R^2 = .13, ΔR^2 = .09)			
Line-length acuity	–0.11	.219	.01
ANS acuity	0.11	.201	.01
Flanker-task performance	0.00	.997	.00
RPS acuity composite	0.34	< .001	.09

Note: n = 151. The ratio of the squared semipartial correlation coefficient (sr^2) for ratio-processing system (RPS) acuity to the summed sr^2 for all control variables was 4.75. ANS = approximate number system.

Why conclude that RPS abilities are antecedents of mathematical ability instead of vice versa? RPS acuity in trained rhesus monkeys approximates that of college undergraduates (Vallentin & Nieder, 2008), which suggests it is a primitive ability, existing prior to instruction. However, there are no baseline studies of rhesus RPS prior to training. Ultimately, we can draw no firm conclusions about the relations between the RPS and education without additional investigation of the time course of RPS development and its relation to mathematical abilities.

Second, the RPS potentially extends accounts of the nonsymbolic foundations of number to include not only fractions, but also any number that can be presented as the ratio between two continuous quantities. Contrary to accounts that intuitive access to numbers ends at whole numbers, our results and others show that humans intuitively process magnitudes that correspond to nonsymbolic ratios, such as those made of line segments. Because line segments can be made arbitrarily long, such ratios can be made corresponding to any positive real number. The formal definition of π is the ratio between a circle's circumference and its diameter. Unrolling a circumference and placing it next to the diameter yields the same sort of ratio used in this experiment.

Third, the RPS operates largely independently of the ANS as currently conceived. We demonstrated that the predictive power of RPS acuity was independent of—and far greater than—that of the ANS for four symbolic outcomes. Moreover, RPS acuity was measured effectively using ratios of line segments, even though line segments should not have engaged the ANS. Observed RPS effects also contrast with recent findings suggesting that the link between ANS and symbolic math ability fades as children grow older (Fazio et al., 2014). On balance, it is clear that nonsymbolic RPS-based abilities are operative in multiple formats and cannot be considered simply subsidiary to the ANS.

Finally, focusing on the RPS may help enrich theories of how symbolic numerical abilities develop. Currently, some researchers speculate that the ANS supports the acquisition of numerical concepts. Part of the argument's appeal lies in the fact that there is a one-to-one map between discrete numerosities and the counting numbers. However, it seems the same systems that estimate absolute magnitude similarly process ratio magnitude (Jacob et al., 2012). To the extent that the RPS can be shown to interface with and influence symbolic number processing, it follows that the expansive psychophysical apparatus can influence numerical development. This possibility would firmly situate numerical development within the more generalized sense of magnitude postulated by Walsh (2003). Future work investigating commonalities in general magnitude perception and explicitly numerical cognition will be necessary to delimit these relations and their practical implications.

Conclusion

Our findings indicate that individual differences in the RPS—a basic nonsymbolic processing capacity that until recently has received little study—predict outcomes on three different measures of symbolic fraction ability and performance on algebra entrance exams at a major selective university. This stands in stark contrast to the assertion that fraction concepts are unsupported by primitive architectures. Indeed, the RPS may prove to be an underappreciated neurocognitive start-up tool that serves as a primitive ground for advanced mathematical concepts. With this in mind, we suggest that the time has come for the field to focus on sensitivity to ratio magnitudes: Instead of relegating ratio to the background as some parameter that influences comparison tasks, perhaps we should elevate ratio sensitivity to the foreground.

Author Contributions

P. G. Matthews, M. R. Lewis, and E. M. Hubbard conceived and designed the study. M. R. Lewis programmed the stimulus presentation software. Testing and data collection were performed by personnel from the labs of P. G. Matthews and E. M. Hubbard, including M. R. Lewis. M. R. Lewis and P. G. Matthews analyzed and interpreted the data. P. G. Matthews, M. R. Lewis, and E. M. Hubbard drafted the manuscript. All authors approved the final version of the manuscript for submission.

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The authors declared that they had no conflicts of interest with respect to their authorship or the publication of this article.

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Open Practices

Access to the data and more information about the experimental stimuli are available from the first author at pmatthews@wisc.edu. The complete Open Practices Disclosure for this article can be found at http://pss.sagepub.com/content/by/supplemental-data.

Note

1. For insight into the richness of the debate beyond this admittedly oversimplified introductory statement, see commentaries on Rips et al. (2008).

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