Target article by Tali Leibovich, Naama Katzin, Maayan Harel, & Avishai Henik

Word counts: Abstract – 52 words, Main Text – 1000, References – 428, Entire Text – 1827

From continuous magnitudes to symbolic numbers: The centrality of ratio

Authors* and Institutions:

Pooja G. Sidney, Kent State University

Clarissa A. Thompson, Kent State University

Percival G. Matthews, University of Wisconsin-Madison

Edward M. Hubbard, University of Wisconsin-Madison

*All authors contributed equally to this paper.

Mailing addresses, telephone numbers, email addresses, and home page URLs:

Pooja G. Sidney, Kent State University, 345 Kent Hall, P.O. Box 5190, Kent, OH 44242, Email: psidney1@kent.edu.

Clarissa A. Thompson, Kent State University, 228 Kent Hall Addition, P.O. Box 5190, Kent, OH 44242, Phone: (330) 672-3948, Email: cthomp77@kent.edu, www.clarissathompson.com.

Percival G Matthews, University of Wisconsin – Madison, 1025 W. Johnson Street #884, Madison, Wisconsin 53706-1796, Phone: (608) 263-3600, Email: pmatthews@wisc.edu, https://website.education.wisc.edu/pmatthews/.

Edward M. Hubbard, University of Wisconsin – Madison, 1025 W. Johnson Street #1075F, Madison, Wisconsin 53706-1796, Phone: (608) 2635-2607, Email: emhubbard@wisc.edu, http://website.education.wisc.edu/edneurolab/

Author Note

Support for this research was provided in part by U.S. Department of Education Institute of Education Sciences: grant R305A160295, National Institutes of Health: grant 1R03HD081087-01, National Science Foundation: grant no. DRL-1419973, and the Kent State University Postdoctoral Seed Grant Program.

Abstract

Leibovich et al.'s theory neither accounts for the deep connections between whole numbers and other classes of number, nor provides a potential mechanism for mapping continuous magnitudes to symbolic numbers. We argue that focusing on nonsymbolic ratio processing abilities can furnish a more expansive account of numerical cognition that remedies these shortcomings.

This commentary was motivated by two shortcomings of the target article by Leibovich et al.: First, its sole focus on whole numbers leaves out entire classes of numbers, such as fractions, that are integral to cultivating robust numerical understanding among children and adults (Siegler, Thompson, & Schneider, 2011). Second, it does not offer a mechanism whereby continuous magnitudes can be linked to specific whole numbers. Below, we argue that focusing on nonsymbolic ratio processing abilities might furnish a more expansive account of numerical cognition, providing perceptual access both to whole number and fraction magnitudes. Moreover, a ratio-focused account can provide a potential mechanism for mapping analog representations of continuous magnitudes to symbolic numbers.

Recent research has begun to systematically detail the ability of humans and other animals to perceive nonsymbolic ratios (e.g., Jacob, Vallentin, & Nieder, 2012; Matthews, Lewis, & Hubbard, 2016; McCrink & Wynn, 2007). Instead of focusing on individual nonsymbolic stimuli in isolation, this work focuses on perceiving ratio magnitudes that emerge from pairs of these stimuli considered in tandem (Figure 1a). As the extent of nonsymbolic ratio processing abilities becomes clearer, some have called for research that foregrounds ratio perception as a possible basis for numerical cognition more generally (e.g., Matthews et al., 2016). Indeed, in a recent book chapter, Leibovich, Kallai, and Itamar (2016) posited that the development of nonsymbolic ratio perception "might be at the background of all other numerical developmental processes…" (p370). In recognition of this fact, we view this commentary as an extension of the authors' own logic to address key gaps in the theory as presented in the target article.

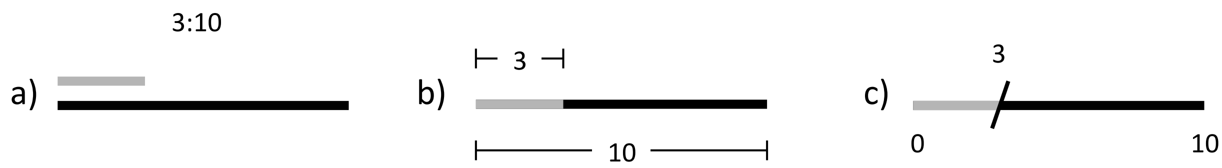

Figure 1. Demonstration of the similarities between nonsymbolic ratios made of line segments and number lines. From left to right, the panels represent (a) an example of a nonsymbolic representation of the ratio 3:10 (or 10:3) based on stimuli from Vallentin & Nieder (2008), (b) the superimposition of the component stimuli of the ratio onto one line, and (c) how the addition of symbolic anchors yields the traditional number line estimation task. At a minimum, accurate number line estimation requires cross-format proportional reasoning, matching the symbolic 3/10 to a corresponding nonsymbolic ratio.

First, we argue that whole numbers are not the whole story. In presenting their integrated theory of numerical development, Siegler et al. (2011) lamented that the field's focus on whole numbers has deflected attention from commonalities shared by both whole numbers and fractions. This is a particularly interesting point given that recent research has highlighted multiple commonalities in the ways we process different classes of number. To name a few:

1. Whole numbers and fractions have both been associated with size congruity effects (Henik & Tzelgov, 1982; Matthews & Lewis, 2016).
2. Processing whole numbers and fractions both recruit the intraparietal sulcus (IPS) (Jacob et al., 2012; Piazza, 2010).
3. Whole numbers and fractions can both be represented as magnitudes on number lines (e.g., Siegler et al., 2011).

4. Processing fractions and whole numbers exhibits distance effects in both symbolic (DeWolf, Grounds, Bassok, & Holyoak, 2014; Moyer & Landauer, 1967) and nonsymbolic forms (Halberda & Feigenson, 2008; Jacob & Nieder, 2009).

This last fact results because numerical processing obeys Weber's Law, and this has two very important implications. The first was perhaps stated best when Moyer and Landauer (1967) wrote that observed distance effects for symbolic numbers implied that it "is conceivable that [numerical] judgments are made in the same way as judgments of stimuli varying along physical continua" (p1520). The second is a corollary to the first and seems widely unappreciated: Weber's Law is fundamentally parameterized in terms of ratios between stimulus magnitudes. Ironically, even the way we represent whole numbers is governed by the ratios among them. Together, these points raise considerable potential for integrating the psychophysics of perception with numerical processing via the conduit of ratio.

Furthermore, we argue that nonsymbolic ratio lays the foundation for a pathway to understanding all real numbers. Leibovich et al.'s theory in the target article bears interesting parallels with Gallistel and Gelman's (2000) theory that the primitive machinery for representing number works with real number magnitudes. The missing link for both is a compelling mechanism for establishing a correspondence between continuous nonsymbolic magnitudes and specific number values. Herein lies the power of nonsymbolic ratios. By juxtaposing two quantities instead of one, ratios of nonsymbolic stimuli can be used to indicate specific values. Although neither the gray nor the black line segments presented in Figure 1a corresponds to a specific number, the ratio between the two corresponds only to 3/10 (or 10/3). Thus, nonsymbolic ratio provides perceptual access to both fractions AND whole numbers. In fact, because the components are continuous, these nonsymbolic ratios can be used to represent *all real numbers*. In this way, nonsymbolic ratios provide a flexible route for mapping non-numerical stimuli to specific real number values.

The potential of this conceptualization becomes clearest when we consider that competent number line estimation (i.e., linear estimates) can be seen as a task bridging symbolic and nonsymbolic proportional reasoning (e.g., Barth & Paladino, 2011; Matthews & Hubbard, in press). Indeed, Thompson and Opfer's (2010) use of progressive alignment with number lines to improve children's symbolic number knowledge can be interpreted as a case in which nonsymbolic ratio perception is used to facilitate analogical mapping that endows unfamiliar symbolic numbers with semantic meaning. This technique leverages the fact that 15:100 is the same as 150:1000 in that both are the same proportion of the way across the number line, a fact that can help children understand the way the base-10 system scales up. Given that nonsymbolic ratio perception is abstract enough even to support comparisons between ratios composed of different types of stimuli (e.g., Matthews & Chesney, 2015, Figure 2), the possibilities for such analogical mapping abound. It may be that much of the psychophysical apparatus that operates in accord with Weber's Law can be used to ground numerical intuitions. A focus on ratio processing stands to firmly situate numerical development within the generalized magnitude system proposed by the target article.

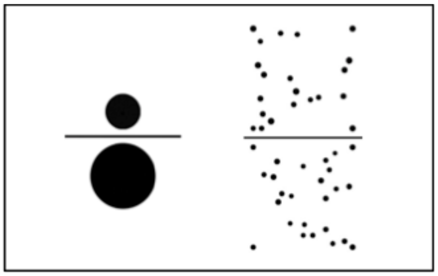

Figure 2. Matthews and Chesney (2015) found that participants could accurately compare nonsymbolic ratios across different formats in about 1100ms – even faster than they could compare pairs of symbolic fractions. This ability to compare ratios across formats implies that participants could perceptually extract abstract ratio magnitudes in an analog form.

A comprehensive theory of numerical development should account for the deep connections between whole numbers and other classes of number, while accounting for relationships between symbolic and nonsymbolic instantiations of numerical magnitudes. Leibovich's theory as presented in the target article neither accounts for numbers like fractions nor accounts for how continuous magnitudes can be mapped to specific numbers. However, adding a correction carving out a pivotal role for nonsymbolic ratio perception might help provide the basis for a unified theory of numerical cognition.

References

Barth, H. C., & Paladino, A. M. (2011). The development of numerical estimation: evidence against a representational shift. *Developmental Science*, *14*(1), 125–135. https://doi.org/10.1111/j.1467-7687.2010.00962.x

DeWolf, M., Grounds, M. A., Bassok, M., & Holyoak, K. J. (2014). Magnitude comparison with different types of rational numbers. *Journal of Experimental Psychology: Human Perception and Performance*, *40*(1), 71–82. https://doi.org/10.1037/a0032916

Gallistel, C. R., & Gelman, R. (2000). Non-verbal numerical cognition: from reals to integers. *Trends in Cognitive Sciences*, *4*(2), 59–65. https://doi.org/10.1016/S1364-6613(99)01424-2

Halberda, J., & Feigenson, L. (2008). Developmental change in the acuity of the "number sense": The approximate number system in 3-, 4-, 5-, and 6-year-olds and adults. *Developmental Psychology*, *44*(5), 1457–1465. https://doi.org/10.1037/a0012682

Henik, A., & Tzelgov, J. (1982). Is three greater than five: The relation between physical and semantic size in comparison tasks. *Memory & Cognition*, *10*(4), 389–395. https://doi.org/10.3758/BF03202431

Jacob, S. N., & Nieder, A. (2009). Tuning to non-symbolic proportions in the human frontoparietal cortex. *European Journal of Neuroscience*, *30*(7), 1432–1442. https://doi.org/10.1111/j.1460-9568.2009.06932.x

Jacob, S. N., Vallentin, D., & Nieder, A. (2012). Relating magnitudes: the brain's code for proportions. *Trends in Cognitive Sciences*, *16*(3), 157–166. https://doi.org/10.1016/j.tics.2012.02.002

Leibovich, T., Kallai, A.Y., & Itamar, S. (2016). What do we measure when we measure magnitudes?” In A. Henik (Ed.), *Continuous Issues in Numerical Cognition*. San Diego: Academic Press.

Matthews, P. G., & Chesney, D. L. (2015). Fractions as percepts? Exploring cross-format distance effects for fractional magnitudes. *Cognitive Psychology*, *78*, 28–56. https://doi.org/10.1016/j.cogpsych.2015.01.006

Matthews, P. G., & Lewis, M. R. (2016). Fractions We Cannot Ignore: The Nonsymbolic Ratio Congruity Effect. *Cognitive Science*, n/a-n/a. https://doi.org/10.1111/cogs.12419

Matthews, P. G., Lewis, M. R., & Hubbard, E. M. (2016). Individual Differences in Nonsymbolic Ratio Processing Predict Symbolic Math Performance. *Psychological Science*, *27*(2), 191–202. https://doi.org/10.1177/0956797615617799

McCrink, K., & Wynn, K. (2007). Ratio Abstraction by 6-Month-Old Infants. *Psychological Science*, *18*(8), 740–745. https://doi.org/10.1111/j.1467-9280.2007.01969.x

Moyer, R. S., & Landauer, T. K. (1967). Time required for Judgements of Numerical Inequality. *, Published Online: 30 September 1967; | doi:10.1038/2151519a0*, *215*(5109), 1519–1520. https://doi.org/10.1038/2151519a0

Piazza, M. (2010). Neurocognitive start-up tools for symbolic number representations. *Trends in Cognitive Sciences*, *14*(12), 542–551. https://doi.org/10.1016/j.tics.2010.09.008

Siegler, R. S., Thompson, C. A., & Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive Psychology*, *62*(4), 273–296. https://doi.org/10.1016/j.cogpsych.2011.03.001

Thompson, C. A., & Opfer, J. E. (2010). How 15 Hundred Is Like 15 Cherries: Effect of Progressive Alignment on Representational Changes in Numerical Cognition. *Child Development*, *81*(6), 1768–1786. https://doi.org/10.1111/j.1467-8624.2010.01509.x

Vallentin, D., & Nieder, A. (2008). Behavioral and Prefrontal Representation of Spatial Proportions in the Monkey. *Current Biology*, *18*(18), 1420–1425. https://doi.org/10.1016/j.cub.2008.08.042