Making Space for Spatial Proportions

Percival G. Matthews and Edward M. Hubbard

pgmatthews@wisc.edu
emhubbard@wisc.edu

Department of Education of Psychology
University of Wisconsin-Madison
1025 W. Johnson St.
Madison, WI 53706

Knowledge of fractions serves as a cornerstone for general mathematical competence (e.g., Siegler et al., 2012). As such, there is a pressing need for work providing insight into the development of fraction knowledge. The three papers presented in this special issue specifically contribute to our understanding of the development of fraction knowledge among children with mathematics learning difficulties. We were excited to see that all three papers converging on the theme of importance of fraction magnitude representations.

Similar to our recent work, these papers suggest that the ability to map between symbolic fractions (like $^{1}/_{5}$) and nonsymbolic spatial representations of their sizes or *magnitudes* (see Figure 1) may be especially important for building robust fraction knowledge. Moreover, it

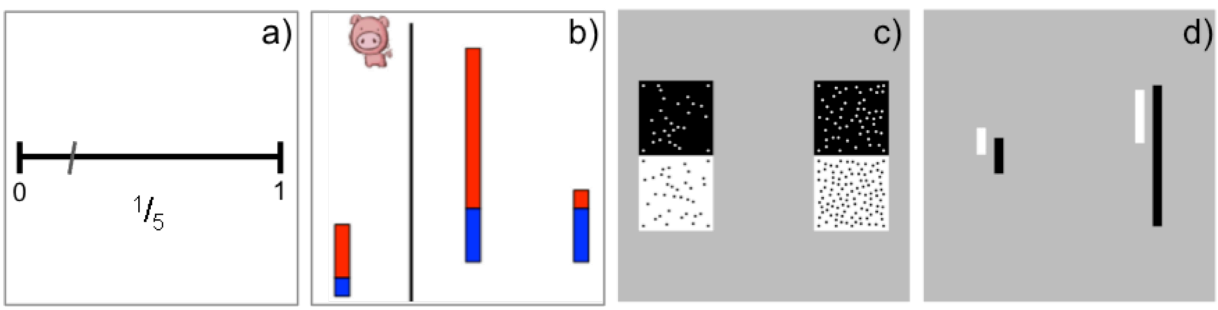

Figure 1. Sample items involving spatial proportional reasoning, including (a) number line estimation, (b) visuospatial matching tasks from Jordan et al. (this issue), and (c) Nonsymbolic ratio comparison tasks (taken from Matthews, Lewis and Hubbard (2015). Despite the diversity of tasks, all involve reasoning about spatial representations of fraction magnitudes.

appears that students with mathematics learning difficulties have characteristic difficulties with tasks requiring these mappings. This finding held true for number line estimation (Tian & Siegler, this issue; Fuchs et al, this issue) and for purely spatial proportional reasoning tasks (Jordan et al, this issue). Below, we first reflect upon where these findings stand in a larger theoretical context, largely borrowed from mathematics education. Next, we emphasize parallels between this work and emerging work suggesting that the ability to perceptually compare ratios may provide an intuitive foundation for understanding fraction magnitudes and that explicitly targeting these abilities may prove to be powerful way to improve fractions knowledge. Finally, we explore a few open questions that suggest specific future directions in this burgeoning area.

Theory in Context – The Fractions 'mega-construct'

Several researchers have argued that fractions – and the rational numbers they represent – are so multifaceted that they should be thought of as a *mega-construct* composed of several different alternative interpretations, or subconstructs (e.g., Behr, Lesh, Post & Silver, 1983; Kieren, 1976, Ohlsson, 1988). Fractions might be interpreted as part/whole comparisons (3 parts out of 4), as indicated divisions (3/4 is the same as 3 divided by 4), as measures on a number line, and as ratios, among other things (Behr et al., 1983). These subconstructs comprise "many related but only partially overlapping ideas" (Ohlsson, 1996, p. 53), and a robust understanding of fractions involves not only understanding these subconstructs, but also how they are interrelated. For instance, the interpretation of fractions as indicated division would predict the frequently observed relationship between knowledge of divisions and fractions knowledge (e.g. Jordan et al., this issue; Siegler et al., 2012). On this view, the dominant focus on the part-whole interpretation in American classroom instruction ignores fundamental aspects of rational number knowledge, and student understanding suffers as a result (e.g., Carraher, 1996).

The findings discussed in the three articles in this special issue seem to confirm that other subconstructs beyond the dominant part-whole interpretation are critical to the formation of fractions knowledge. In particular, consistent with arguments we have recently made (Lewis, Matthews & Hubbard, 2015; Matthews, Lewis & Hubbard, in press) each highlights the heretofore underappreciated role that spatial proportional reasoning plays in supporting fractions concepts. Indeed, Fuchs et al (this issue) found that an intervention that focused on the measurement interpretation of fractions, and not the part-whole interpretation alone, led to improvements in scores on a set of fractions items from the NAEP. Improvements in understanding fractions as measurements led to improvements in the part-whole understanding of fractions, but improvements in part-whole understanding did not lead to increased understanding of the measurement interpretation.

The Importance of Spatial Proportional Reasoning

In reflecting on the nature of the number line estimation task so essential to the work of Tian and Siegler, we recall Siegler and Booth's (2005) definition of estimation as "the process of translating between alternative quantitative representations, at least one of which is inexact" (p. 198). For number lines, the number symbols involved obviously serve as exact representations. However, it is the inexact representational component of number line estimation – the one that involves mapping number symbols to space – that is most interesting. Because any given number line can be arbitrarily long, finding the proper spot to mark relies on proportional reasoning: It requires estimating the magnitude of one stimulus, the to-be-placed value, *relative* to another stimulus, the distance between the upper and lower endpoints of the number line. In this sense, number line estimation tasks are a type of cross-format proportional reasoning.[1]

This interpretation of the number line estimation reveals parallels between Tian and Siegler's work and recent findings by Möhring, Newcombe, Levine and Frick (2015) suggesting that spatial proportional reasoning is associated with formal knowledge about fractions. However striking it is that facility with spatial proportions is generally related to fractions knowledge, it is at least as intriguing that individual differences in nonsymbolic spatial proportional reasoning may prove to be predictive of math learning difficulties. Notably, Tian and Siegler found that differences in number line estimation could predict learning difficulties among pooled American

[1] We make this point without choosing a side in the discussion about whether number line estimation involves a cyclical power model, as debated most prominently by Opfer, Siegler and Young (2011) and Barth, Slusser, Cohen and Paladino (2011).

and Chinese samples, whereas fraction arithmetic items could not be used to distinguish Chinese students with learning difficulties from typically achieving US students.

Jordan and colleagues similarly document the importance of spatial proportional reasoning, both among 3rd graders and among 5th graders. Although the most powerful predictor of learning difficulties amongst third graders was number line estimation, the interpretation we outlined above also suggests that even these tasks involve proportional reasoning across symbolic and nonsymbolic contexts. Thus, even placing 7 on a number line from 0-10 is in some sense a proportional reasoning task involving the ratio 7:10 in symbolic and spatial formats. On this view, number line estimation – regardless of the class of number involved – might be expected to predict symbolic fraction knowledge, because number line estimation by its very nature implicitly entails proportional reasoning.

The predictive power of purely nonsymbolic proportional reasoning tasks for 5th graders is more interesting and might seem counterintuitive to some. Whereas number line estimation involves mapping between symbolic and nonsymbolic spatial representations of magnitude, the nonsymbolic proportions used by Jordan et al. did not involve number symbols at all. Instead their task involved equating different instantiations of inexact spatial ratio magnitudes. Still, it remained a powerful predictor, explaining unique variance even beyond that explained by number line estimation. Indeed, this purely spatial task explained more of the variance in fractions knowledge than did long division with number symbols. The authors conclude that the visuospatial system provides a sense of proportion that helps students reason about fraction magnitudes in ways that verbally mediated knowledge does not (see also Lewis et al., 2015; Möhring et al., 2015). This point is part of an emerging viewpoint that holds significant potential for changing conceptions of how the human cognitive architecture comes to access fraction magnitudes.

A Ratio Processing System?

Tian and Siegler and Jordan and colleagues highlight the explanatory power of tasks tapping the ability to accurately complete proportions involving spatially instantiated ratios. This insight directly parallels recent work suggesting that (1) evolution has endowed humans with a perceptually-based ratio processing system that is sensitive to magnitudes of nonsymbolic ratios (Jacob, Vallentin, & Nieder, 2012); and (2) that the RPS may be ideally suited for grounding fraction concepts (Lewis et al., 2015; Matthews et al., in press). In our own recent work, we have suggested that RPS-based processing supports learners' understanding of the overall magnitude of symbolic fractions, even when formal instruction does not explicitly attempt to leverage it. Our theory predicts that both formal and informal learning experiences help to generate links among symbolic fractions and RPS representations of nonsymbolic ratio magnitudes that help make fractions symbols meaningful (Lewis et al., 2015). It also predicts that individual differences in RPS acuity promote better fraction knowledge, and the findings in this series of papers are consistent with that hypothesis. Given the importance of fraction knowledge in mathematical development, we therefore predicted that RPS acuity should predict mathematical outcomes more generally.

To directly test this prediction, we used nonsymbolic ratio comparison tasks that are quite similar to the nonsymbolic proportion tasks employed by Jordan and colleagues. First, we measured adult's RPS acuity by asking them to judge which of two nonsymbolic ratios (instantiated using either line segments or dot arrays) had the larger magnitude. Then we investigated whether individual differences in RPS acuity could explain unique variance for performance on symbolic fraction comparison tasks, number line estimation with fractions, a general test of fraction knowledge, and algebra entrance exam scores at a large selective university. We found that RPS acuity predicted performance on all four symbolic outcomes, even when controlling for other non-symbolic comparison abilities. Our findings directly parallel those of Jordan et al., although our study tested an older sample and used a more distal outcome measure (algebra). Moreover, they directly confirm the relations between purely nonsymbolic proportional reasoning and number line estimation, which Tian and Siegler found to be predictive of mathematics learning difficulties. Viewed together, this emerging body of findings is consistent with the hypothesis that the ability to process nonsymbolic ratio magnitudes may be an important proto-mathematical ability upon which emerging mathematical competence is built (Lewis et al., 2015; Matthews et al., 2015; Möhring et al., 2015).

Future Directions

Our readings of the three papers motivate us to propose two sets of future directions. The first set involves basic science regarding the overall predictive power for spatial proportional reasoning and the course of its development. Future studies should examine how these abilities change over development, and whether individual differences are predictive of outcomes over larger swaths of developmental time than the single year-to-year prediction shown by Jordan and colleagues. Might individual differences in spatial proportional reasoning emerge as a diagnostic factor for assessing mathematics learning difficulties or disabilities? Additionally, it would be important to know how spatial proportional reasoning abilities relate to the development of number line estimation abilities. It is striking that this purely nonsymbolic ability should have such a strong relations to symbolic math competence, and seeking the roots of these relations is an important line of inquiry.

The second set of future directions is more practical in nature. Currently, nonsymbolic proportional reasoning is not a pedagogical focus of major curricula. However, it is that this factor exerts its effects remains indirect from the perspective of popular instruction. Nevertheless, Fuchs and colleagues have demonstrated that curricula that explicitly focus on number line estimation can be helpful for remediating mathematics learning difficulties. Might more explicit focus on spatial proportional reasoning per se prove to be similarly productive? Are there other nonsymbolic formats that might further augment the beneficial effects of using number lines? Ultimately, it is possible that explicit focus on spatial proportional reasoning might help ameliorate the very difficulties that emerge when those spatial proportion abilities are taught using typical Tier 1 instruction. Indeed, we know that explicit focus on phonology improves reading outcomes for children with poor phonological awareness (Shaywitz et al., 2004), and the hope is that the fruits of this current work may eventually prove to be just as powerful.

Barth, H., Slusser, E., Cohen, D., & Paladino, A. (2011). A sense of proportion: Commentary on Opfer, Siegler, and Young. *Developmental Science*, *14*(5), 1205–1206.

Behr, M. J., Lesh, R., Post, T. R., & Silver, E. A. (1983). Rational number concepts. *Acquisition of mathematics concepts and processes*, 91-126.Jacob, S. N., Vallentin, D., & Nieder, A. (2012). Relating magnitudes: the brain's code for proportions. *Trends in Cognitive Sciences*, *16*(3), 157–166. http://doi.org/10.1016/j.tics.2012.02.002

Carraher, D.W.: 1996, 'Learning about fractions,' in L.P. Steffe, P. Nesher, P. Cobb, G.A. Goldin and B. Greer (eds.), *Theories of Mathematical Learning*, Lawrence Erlbaum Associates, New Jersey, pp. 241–266.

Kieren, T. E. (1976). On the Mathematical, Cognitive and Instructional. In *Number and Measurement. Papers from a Research Workshop.* (Vol. 7418491, p. 101). ERIC.

Lewis, M. R., Matthews, P. M., & Hubbard, E. M. (2015). Neurocognitive architectures and the nonsymbolic foundations of fractions understanding. In D. B. Berch, D. C. Geary, & K. M. Koepke (Eds.), *Development of Mathematical Cognition: Neural Substrates and Genetic Influences* (pp. 141–160). San Diego, CA: Academic Press.

Matthews, P. G., Lewis M. R., & Hubbard, E. M. (in press). Individual Differences in Nonsymbolic Ratio Processing Predict Symbolic Math Performance. *Psychological Science.*

Möhring, W., Newcombe, N. S., Levine, S. C., & Frick, A. (2015). Spatial Proportional Reasoning is Associated with Formal Knowledge about Fractions. *Journal of Cognition and Development*, *0*(ja), null. http://doi.org/10.1080/15248372.2014.996289

Ohlsson, S.: 1988, ‘Mathematical meaning and applicational meaning in the semantics of fractions and related concepts’, in J. Hiebert and M. Behr (eds.),*Number Concepts and Operations in the Middle Grades*, NCTM/LEA, Reston, pp. 53–91.

Opfer, J. E., Siegler, R. S., & Young, C. J. (2011). The powers of noise-fitting: reply to Barth and Paladino. *Developmental Science*, *14*(5), 1194–1204; discussion 1205–1206. http://doi.org/10.1111/j.1467-7687.2011.01070.x

Shaywitz, B. A., Shaywitz, S. E., Blachman, B. A., Pugh, K. R., Fulbright, R. K., Skudlarski, P., … Gore, J. C. (2004). Development of left occipitotemporal systems for skilled reading in children after a phonologically- based intervention. *Biological Psychiatry*, *55*(9), 926–933. http://doi.org/10.1016/j.biopsych.2003.12.019

Siegler, R. S., & Booth, J. L. (2005). Development of numerical estimation: A review. *Handbook of Mathematical Cognition*, 197–212.

Siegler, R. S., Duncan, G. J., Davis-Kean, P. E., Duckworth, K., Claessens, A., Engel, M., … Chen, M. (2012). Early Predictors of High School Mathematics Achievement. *Psychological Science*, *23*(7), 691–697. http://doi.org/10.1177/0956797612440101