Individual Differences in Nonsymbolic Ratio Processing Predict Symbolic Math Performance

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Abstract

What basic capacities lay the foundation for advanced numerical cognition? Are there basic nonsymbolic abilities that support understanding more advanced numerical concepts like fractions? To date most theories have posited that previously identified core numerical systems like the approximate number system (ANS), are ill-suited for learning fraction concepts. However, recent research in developmental psychology and neuroscience has revealed a “ratio processing system” (RPS) sensitive to magnitudes of nonsymbolic ratios that may be ideally suited for grounding fractions concepts. We provide evidence for this hypothesis by showing that individual differences in RPS acuity predict performance on four measures of mathematical competence, including a university algebra entrance exam. We suggest that the nonsymbolic RPS may support symbolic fraction understanding much like the ANS supports whole number concepts. Thus, even abstract mathematical concepts like fractions may be grounded as much in basic nonsymbolic processing abilities as they are in higher-order logic and language.

Keywords: fractions; neurocognitive architecture; number sense; nonsymbolic ratios; ratio processing system

What is a number that a man may know it, and a man that he may know a number? – Warren McCulloch

Developmental and cognitive psychologists have long debated McCulloch's question, with some arguing that numerical cognition is constructed in a top-down manner using higher-order cognitive abilities (Piaget, 2013; Rips, Bloomfield, & Asmuth, 2008; Wiese, 2003) whereas others have argued that numerical cognition is grounded in a primitive nonsymbolic "number sense" (Feigenson, Dehaene, & Spelke, 2004; Gallistel & Gelman, 2000; Nieder, 2005).[1] Cognitive primitive accounts have been limited in that they are often deemed inadequate for explaining how humans understand numbers beyond whole numbers, such as fractions or square roots (but see Barth, Starr, & Sullivan, 2009; McCrink, Spelke, Dehaene, & Pica, 2013). Some cognitive primitive theorists explicitly argue that limitations of core numerical systems constrain the number concepts that are intuitively accessible (Feigenson et al., 2004; Wynn, 1995). For instance, Dehaene (2011) concluded that numbers like fractions "defy intuition because they do not correspond to any preexisting category in our brain" (p. 76). Consequently, such accounts have arguably capitulated to logical/linguistic accounts with respect to more advanced numerical concepts: Numbers like fractions are logical constructs that can only be understood with great difficulty by recycling whole number schemas.

Here, we present evidence that cognitive primitive accounts can be expanded to apply to fractions, and perhaps even to all positive real numbers. We demonstrate that individual differences in a hitherto underappreciated nonsymbolic "ratio processing system" (or RPS) predict differences in higher-order math skills, including symbolic fraction competence and algebra achievement scores.

[1] For insight into the richness of the debate beyond this admittedly oversimplified introductory statement, see commentaries on Rips et al. (2008).

‘Neurocognitive Startup Tools’ for Fractions Concepts?

Advances in cognitive and neural sciences have allowed researchers to chart what Piazza (2010) dubbed “neurocognitive startup tools” that provide the foundations of numerical cognition. These include the object tracking system (OTS), which supports rapid exact enumeration of small sets, and the approximate number system (ANS), which enables humans and other animals to quickly approximate numerosities of larger sets. It has been proposed that these basic abilities are “recycled” by formal learning to support acquisition of math concepts with parallel structures (Dehaene & Cohen, 2007; Piazza, 2010; but see De Smedt, Noël, Gilmore, & Ansari, 2013).

However, ANS and OTS abilities ostensibly appear to provide a poor foundation for understanding numbers that are not count-based. For example, fractions like “$^{1}/_{2}$” cannot be reached by counting. Moreover, each fractional magnitude can be represented in infinite ways (e.g., $^{1}/_{2}$ represents the same magnitude as $^{2}/_{4}$, $^{11}/_{22}$, etc.). Consequently, constructing representations of fraction magnitude by recycling ANS- or OTS-based representations may be a necessarily difficult and error prone process (but see DeWolf, Bassok, & Holyoak, 2015). Indeed, both children and highly educated adults often struggle to understand fractions (Lipkus, Samsa, & Rimer, 2001; Ni & Zhou, 2005). Hence, it may initially seem reasonable to conclude that these pervasive difficulties arise from innate constraints on human cognitive architecture (see Ni & Zhou, 2005 for a review).

However, there is no a priori reason to assume that these are the only intuitive building blocks available. Indeed, Jacob, Vallentin and Nieder (2012) have identified dedicated neural networks that have evolved that automatically process nonsymbolic ratio magnitudes in multiple formats. Recent developmental and neuroscientific findings are consistent with such a potential nonsymbolic building block: McCrink & Wynn (2007) showed that 6-month-old infants are sensitive to nonsymbolic ratios and by age 4, children can complete tasks requiring addition and subtraction of nonsymbolic part-whole fractions (Mix, Levine, & Huttenlocher, 1999). Moreover, research with both American 6- and 7- year olds (Barth et al., 2009) and with children from indigenous tribes with limited number concepts (McCrink et al., 2013) can effectively perform multiplicative scaling with numerosities. By age 11, children's performance on number line estimation with nonsymbolic fractions is correlated with standardized test scores (Fazio, Bailey, Thompson, & Siegler, 2014). Finally, past work on psychophysical scaling may also implicate nonsymbolic ratio processing abilities, (e.g., Hollands & Dyre, 2000; Stevens & Galanter, 1957).

This sensitivity to nonsymbolic ratios is flexible and abstract. Singer-Freeman and Goswami (2001) showed that children can draw proportional analogies among pizzas, chocolates and lemonade even though the materials equated are visually dissimilar. Matthews and Chesney (2015) demonstrated that participants made cross-format nonsymbolic ratio comparisons (Fig. 1) faster than they made symbolic fraction comparisons, suggesting that comparisons were made without first translating nonsymbolic ratios into symbolic form. Finally, Vallentin and Nieder (2008) showed that monkeys match nonsymbolic ratios even with substantial variation in the component line lengths. Such flexibility across developmental time, with novel stimuli, and even across species is consistent with the existence of an RPS, which may provide an intuitive foundation for understanding fractions magnitudes.

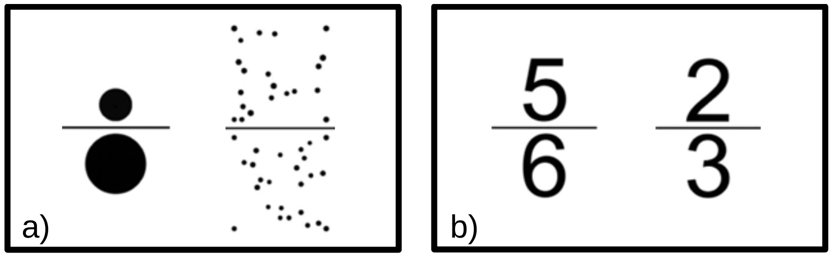

Figure 1. Matthews and Chesney (2015) found that participants completed nonsymbolic ratio comparisons across formats (panel a) more quickly than they completed symbolic fraction comparisons (panel b), suggesting that nonsymbolic ratio comparisons were made without conversion to symbolic form.

Hypothesized RPS Support for Emerging Fractions Concepts

Recent research has shown that understanding fractions magnitudes is a key factor in promoting the acquisition of other fractions concepts and procedures (Siegler et al., 2012; Siegler & Pyke, 2012). We suggest that RPS-based processing supports learners' understanding of the overall magnitude of symbolic fractions, even when formal instruction does not explicitly attempt to leverage it. Our theory (see Lewis, Matthews & Hubbard, in press) specifies that:

1. Both formal and informal learning experiences help to generate links among symbolic fractions and RPS representations of nonsymbolic ratio magnitudes. These links help ground fractions symbols, making them meaningful (compare with Siegler & Lortie-Forgues, 2014). This influences later learning involving fractions concepts.
2. Individual differences in RPS acuity can moderate effects of learning experiences, so learners with better RPS acuity should build more precise symbolic-to-nonsymbolic links, promoting better fractions knowledge.

3. Following Jacob and Nieder's (2009) demonstration that humans process nonsymbolic ratios even when viewed passively, the RPS should exert its effects on learning even when it is not an explicit pedagogical focus.

Here, we tested a key prediction of our account: that individual differences in RPS acuity would predict symbolic mathematical competence.

The Current Study

Despite evidence that humans have sensitivity to nonsymbolic ratio magnitudes, the connections between RPS acuity and symbolic mathematical ability have remained largely uninvestigated (but see Fazio et al., 2014). We used a multiple regression framework to test whether RPS acuity predicts symbolic math skills.

Method

Participants

Participants were 183 undergraduate students at a large American university who participated for course credit. Two participants did not understand directions for the computerized tasks and were excused, yielding the final sample of 181 (155 female; ages M = 19.8, Range = 18–22).

Power analysis revealed that to detect medium-sized effects for our three fractions outcomes (Cohen's f = .15) given four predictors would require 84 participants. We expected effects for predicting the more distal Algebra score to be smaller, but there was no preexisting literature to guide expectations for how much smaller. Hence, we set out to recruit 200 participants. 183 were recruited before the end of the academic year when data collection ended.

Materials and Procedures

For our predictors, we constructed four nonsymbolic ratio comparison tasks that paralleled those typically used to measure ANS acuity. Jacob and Nieder (2009) previously used similar task to assess the neural representation of proportions, but the tasks have never been used in connection with symbolic math outcomes. In each, the participants' task was to choose the larger of two nonsymbolic ratios (Fig. 2). Ratio size was determined by dividing the magnitude of smaller component of a ratio by that of the larger. For instance, for line ratios, the length of the white line segment was divided by the length of the black line segment. Thus, the ratio on the left side of Fig. 2, panel b is larger than that on the right side, despite the fact that the individual components composing the ratio on the right side are larger.

Outcome measures were three measures of symbolic fractions knowledge and a measure of Algebra from participants' university entrance examination. Algebra scores served as a distal measure, because previous research has shown fractions knowledge to be a critical predictor of Algebra performance (Siegler et al., 2012). We also included other cognitive tasks as covariates: Two tasks controlled for participants' abilities to process the absolute magnitudes of the components of nonsymbolic ratios in contrast to their relative magnitudes (i.e., an ANS-based numerosity discrimination task and a line length discrimination task). A flanker task controlled for differences in inhibitory control. Tasks are described in more detail below.

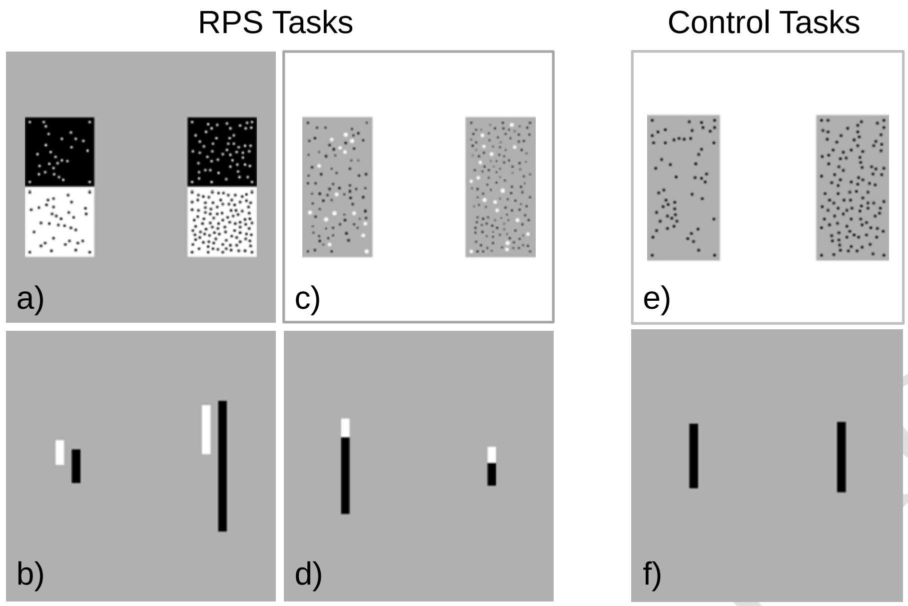

Figure 2. Sample stimuli from the nonsymbolic comparison tasks. Tasks (a) – (d) were predictor tasks measuring RPS acuity in different formats. Tasks (a) and (b) assessed RPS acuity using representations in which the components appear separately, whereas tasks (c) and (d) used representations with integrated components. Tasks (e) and (f) were used to control for individual differences in discrimination acuity for dots and lines using individual stimuli of the same formats.

Nonsymbolic ratio comparisons (RPS acuity tasks). These four tasks assessed the ability to discriminate between nonsymbolic ratio values composed either of dot arrays or line segments. Tasks included separate-component dot ratio comparisons, integrated-component dot ratio comparisons, separate-component line ratio comparisons, and integrated-component line ratio comparisons. We implemented several controls to reduce reliance on the magnitudes of individual components (i.e., any strategic bias to base judgments either on the numerator or the denominator rather than on overall fraction values) and irrelevant surface features in the nonsymbolic ratio comparison tasks (e.g. the sizes of dots in an array as opposed to the number of dots). For half the trials, components of the larger ratio were shorter/less numerous than the corresponding components of the smaller ratio. In the remainder, components of the smaller ratios were shorter/less numerous than the corresponding components of the larger ratio. For dot stimuli, in half of trials sets had the same cumulative area but variable dot sizes, such that average dot area in an array was correlated with numerosity. In the other half, dot size was held constant, such that overall area was correlated with numerosity.

Nonsymbolic control comparisons. These tasks assessed participants' abilities to discriminate between pairs of individual dot arrays (ANS acuity) or pairs of individual line segments. Control comparison tasks allowed us to measure the independent contribution of RPS acuity over and above effects due to acuity for processing pairs of individual stimuli.

For all nonsymbolic comparisons, participants selected the larger nonsymbolic ratio, the more numerous dot array, or the longer line segment via key press (i.e., pressed "j" to indicate that the right stimulus was larger and "f" to indicate the left was larger). Only one type of comparison was performed per block. Each trial began with a fixation cross for 1000ms, immediately followed by brief presentation of two comparison stimuli (Fig. 3). Because each nonsymbolic ratio stimulus was constructed of two components (i.e., one for the numerator and one for the denominator), ratio stimulus pairs were visible for 1500ms before disappearing, whereas control stimuli were visible for 750ms. Trials did not progress until participants submitted a response. There was no time limit imposed, but piloting indicated that responses would be fast (< 2000ms on average).

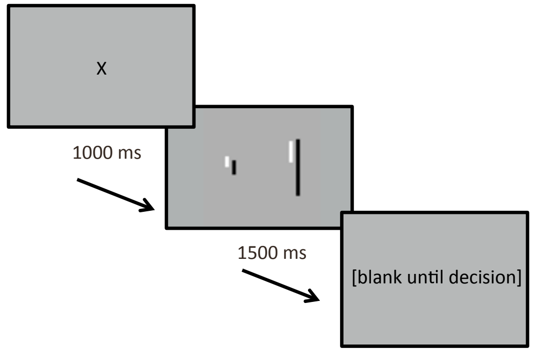

Figure 3. Sample depiction of an integrated line ratio comparison trial. Participants answered by pressing ‘f’ (left) or ‘j’ (right) keys to indicate which stimulus they thought had the larger fractional value. In this case, the ratio on the left was larger, as the white line is almost the same length as the black line.

Each block began with 15 practice trials followed by 40 experimental trials for control blocks and 10 practice trials followed by 45 experimental trials for ratio comparison blocks. Difficulty varied from trial to trial and was operationalized as the ratio between stimulus magnitudes compared in a trial, with difficulty increasing as the ratio approached 1:1. For example, a difficulty ratio of 2:1 might be instantiated by the comparison of 100 dots to 50 dots in a control comparison task. Likewise, it could be instantiated in a dot ratio comparison task by juxtaposing 50 dots over 100 dots in one stimulus and juxtaposing 20 over 80 dots in another (i.e., nonsymbolic instantiations of ½ vs. $^{1}/_{4}$). Difficulty ratios tested varied by task, because pilot experiments indicated that acuity varied from task to task. Based on pilot data, difficulty ratios were chosen to span the spectrum from easy enough to elicit near perfect performance to difficult enough to elicit chance performance at the group level (Table 1). All participants judged the same comparison stimuli, but the order of these stimuli was randomized for each participant.

Table 1

Minimum and Maximum Ratios between Stimulus Pairs in Each Format

	Stimulus Type					
	Dots Control	Dot ratios integrated-components	Dot ratios separate-components	Line Discrimination control	Line ratios separate-components	Line ratios integrated-components
Max ratio	3:1	4:1	3:1	11:10	2:1	2:1
Min ratio	8:7	4:3	6:5	15:14	8:7	8:7

Note: For ratio stimuli, each cell represents a ratio of ratios (e.g., a $^3/_5$ vs. ¾ comparison is a 5:3 ratio)

Symbolic fraction comparison. Participants selected the larger of two symbolic fractions. All fractions were composed of single digit numerators and denominators. To reduce reliance on componential strategies (i.e., any strategic bias to base judgments either on the numerator or the denominator rather than on overall fraction values), fraction pairs shared no common components (i.e., all numerators and denominators in any given pair were unique, as in ¾ vs. $^2/_5$) per Schneider and Siegler (2010). Thirty pairs were sampled at random from all possible combinations of single digit irreducible fractions except ½, because previous work suggests that ½ has a special status (e.g., Spinillo & Bryant, 1991). All participants judged the same fraction pairs, but the order of the fraction pairs was randomized for each participant. Side of presentation was randomized, with the restriction that in half of the trials, the larger fraction was presented on the left.

Each trial began with a fixation cross for 750ms, immediately followed by presentation of the fraction pair until either the participant responded or 5000ms had elapsed, at which point the trial timed out. Prior to experimental trials, participants made 5 practice judgments.

Number line estimation. The number line estimation task was patterned after Siegler, Thompson and Schneider (2011). Participants used a computer mouse to indicate the position of fraction stimuli ($^{1}/_{19}$, $^{1}/_{7}$, $^{1}/_{4}$, $^{3}/_{8}$, $^{1}/_{2}$, $^{4}/_{7}$, $^{2}/_{3}$, $^{7}/_{9}$, $^{5}/_{6}$, or $^{12}/_{13}$) on a number line with end points 0 and 1. Performance was defined as each participants' mean percentage absolute error (PAE), where PAE = (|Answer – Correct Answer|/Numerical Range). A smaller PAE indicates more accurate responding.

Flanker task. Prior work has shown that participants often have to inhibit their initial inclinations to respond based on the sizes of fraction components rather than the overall size of the fraction (DeWolf & Vosniadou, 2015). For example ¼ is larger than $^{1}/_{5}$, but participants are tempted to select $^{1}/_{5}$ as larger, because 5 is larger than 4. Thus, it seems that individual differences in cognitive control may be important for explaining individual differences in fraction processing. We included the flanker task as a covariate to control for any such differences in cognitive control. Each trial began with the presentation of a fixation cross for 500ms, followed by an array of five equally-sized and equally-spaced white arrows on a black background. Arrays remained on-screen for 800ms or until participants made a response. Participants were instructed to "choose which direction the center arrow is pointing." Participants first completed 12 practice trials and then 80 experimental trials. Forty of these trials were "congruent", with the four flanking arrows pointed in the same direction as the center arrow (e.g., "> > > > >"). The other 40 trials were "incongruent", with the four flanking arrows pointed in the opposite direction (e.g. "< < > < <"). In half of the trials, the center arrow pointed left, and in the other half it pointed right. The order of trials was randomized for each participant. Trials in which no response was provided within 800ms were scored as incorrect. Accuracy was used as the independent variable in the analysis below.

Fraction knowledge assessment (FKA). The FKA is a 38-item pencil-and-paper assessment of both procedural and conceptual fraction knowledge. We constructed this instrument using items taken from key national and international assessments including the NAEP and TIMSS and from instruments developed by psychology and math education researchers (Carpenter, 1981; Hallett, Nunes, Bryant, & Thorpe, 2012). The FKA had strong internal consistency (Cronbach's $\alpha = 0.88$).

Algebra entrance exam. The algebra entrance exam is a 35-item subtest of the university placement exam taken by all incoming freshmen. The algebra subtest has strong internal consistency (Cronbach's $\alpha = 0.90$). The assessment was developed with scale scores normed to a mean of 500 and a standard deviation of 100. Note that analysis of the relations between RPS and algebra scores were confined to the portion of the sample who enrolled in the university system as freshmen, because exam scores were not available for 24 participants who transferred into the university after first attending other colleges.

Outlier removal and recoded cases. For all comparison tasks (symbolic and nonsymbolic), we first excluded data with RTs below 250ms and with RTs greater than 5000ms (to accord with the 5000ms limit imposed upon symbolic fraction comparisons). Next, at the participant level, we removed trials with RTs that were more than 3 standard deviations faster or slower than a participant's mean response time for that task. Altogether, these steps resulted in the loss of 1.5-3.2% of the data for each task. Finally, for tasks other than the FKA and Algebra entrance exam, we excluded data from participants who scored more than 3 standard deviations from the group mean for that task, while including those participants' data from other tasks. This resulted in the exclusion of 1 participant form the separate-component line ratio comparisons, 1 from the ANS control, 2 from the line segment comparison control, 3 from number line estimation, and 5 from the flanker task.

In a few cases, participants appeared to give reversed responses. For example, two participants in the flanker task had less than 20 percent accuracy, whereas average accuracy was 92.8% (*SD* = 9.3%). These participants' responses were reverse coded prior to analysis (2 participants for separate-component line ratios, 5 for separate-component dot ratios, and 2 for the flanker task). We also completed supplemental analyses without this recoding. All significant effects for RPS acuity reported below remained significant when analyses were re-run without recoding.

Results

RPS Acuity

Although the nonsymbolic ratio comparison tasks were novel, participants were able to effectively complete the tasks; average accuracy for each task was 70% or higher. Moreover, accuracy was significantly correlated across different RPS tasks (Table 2). We conducted an exploratory factor analysis to assess whether these separate tasks loaded on the same latent RPS construct despite differences in format. The analysis yielded a single factor solution (eigenvalue = 2.24). No other factors had eigenvalues greater than one, suggesting that all four tasks measured the same factor, RPS acuity. We therefore constructed a composite RPS acuity score, defined as accuracy averaged across all four nonsymbolic ratio comparison tasks for use in the analyses that follow.

Table 2

Bivariate Correlations Among Nonsymbolic Ratio Comparison Measures

Task	2	3	4	5
1. Lines_Separate	.44**	.47**	.32**	.71**
2. Lines_Integrated		.46**	.43**	.79**
3. Dots_Separate			.35**	.78**
4. Dots_Integrated				.71**
5. RPS Composite				

** $p < .001$

Predicting Symbolic Math from RPS Acuity

We conducted a series of two-stage hierarchical linear regressions to test whether individual differences in composite RPS acuity predicted symbolic fraction knowledge and algebra achievement above and beyond individual differences in control measures of ANS acuity, line length acuity, and flanker performance (see Fig. 4 for scatterplots based on zero-order correlations). In the first stage of each analysis, we entered the control variables *ANS acuity*, *line discrimination*, and *flanke*r to yield a base model. In the second stage, we added *composite RPS acuity*. Sample sizes and regression results are indicated in Tables 3 – 6. We report standardized coefficients as they facilitate easy interpretation of effect sizes that parallel Cohen's *D*. Note that the outlier removal in addition to the missing FKA scores for transfer students led to different numbers of data points for regressions on different outcomes, as indicated in the tables.

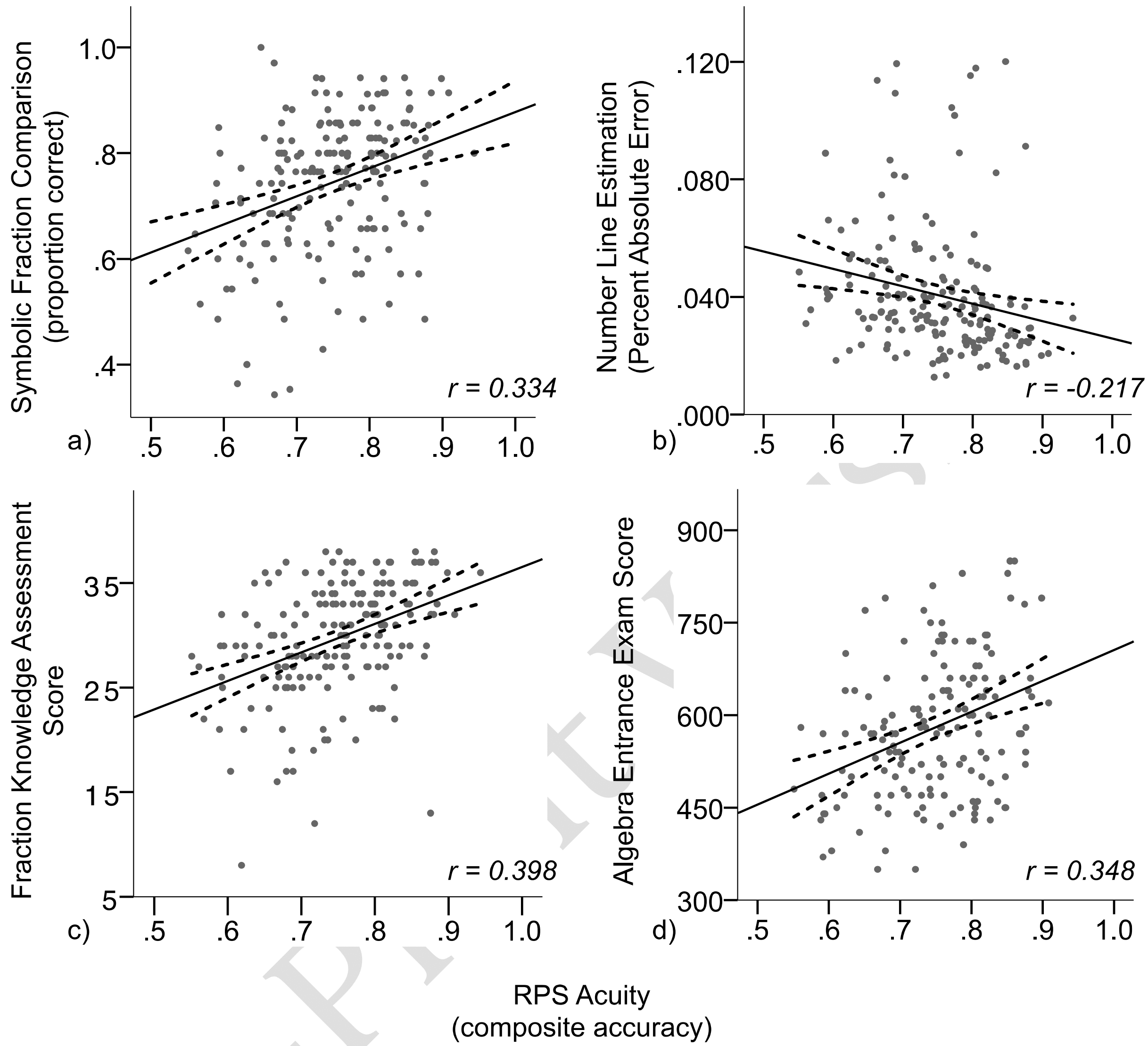

Figure 4. Scatterplots depicting zero-order correlations between RPS acuity and (a) symbolic fraction comparison, (b) number line estimation, (c) fractions knowledge assessment scores, and (d) Algebra entrance exam scores. The x-axis for each graph is accuracy averaged across the four RPS tasks. Note that a lower PAE denotes better performance in panel (b).

Stage 1: Control measures. Performance on line segment comparisons and on flanker tasks failed to significantly predict performance for any of the four symbolic mathematics

Table 3

Hierarchical Regression Analysis Predicting Symbolic Fraction Comparison Accuracy

Variable	β	*p*	sr^2	R^2	ΔR^2
Step 1				.03	.03
Line Length	0.01	0.906	.00		
ANS Acuity	0.16	0.053	.02		
Flanker	-0.04	0.573	.00		
Step 2				.14	.11
Line Length	-0.09	.285	.01		
ANS Acuity	0.08	.305	.01		
Flanker	-0.12	.111	.01		
RPS Composite	0.38	< .001	.11		
n = 174			$sr^2_{RPS}/sr^2_{controls} = 4.57$		

Note: All regression coefficients are standardized for Tables 3-6.

Table 4

Hierarchical Regression Analysis Predicting Fraction Number Line PAE

Variable	β	*p*	Sr^2	R^2	ΔR^2
Step 1				.03	.03
Line Length	-0.13	.105	.02		
ANS Acuity	-0.05	.516	.00		
Flanker	0.00	.981	.00		
Step 2				.07	.04
Line Length	-0.08	.351	.00		
ANS Acuity	-0.01	.937	.00		
Flanker	0.05	.565	.00		
RPS Composite	-0.22	.009	.04		
n = 172			$sr^2_{RPS}/sr^2_{controls} = 5.72$		

Table 5

Hierarchical Regression Analysis Predicting Fraction Knowledge Assessment Scores

Variable	β	*p*	Sr^2	R^2	ΔR^2
Step 1				.05	.05
Line Length	0.10	0.229	.01		
ANS Acuity	0.18	0.033	.03		
Flanker	-0.02	0.779	.00		
Step 2				.20	.15
Line Length	-0.01	.881	.00		
ANS Acuity	0.08	.293	.00		
Flanker	-0.11	.151	.01		
RPS Composite	00.43	< .001	.15		
n = 171			$sr^2_{RPS}/sr^2_{controls} = 9.90$		

Table 6

Hierarchical Regression Analysis Predicting Algebra Entrance Exam Scores

Variable	β	*p*	Sr^2	R^2	ΔR^2
Step 1				.04	.04
Line Length	-0.03	0.774	.00		
ANS Acuity	0.17	0.050	.03		
Flanker	0.08	0.333	.01		
Step 2				.13	.09
Line Length	-0.11	0.219	.01		
ANS Acuity	0.11	.201	.01		
Flanker	0.00	.997	.00		
RPS Composite	0.34	< .001	.09		
n = 151			$sr^2_{RPS}/sr^2_{controls} = 4.75$		

outcome measures (ps ≥ .10). However, ANS acuity was a significant predictor for both FKA performance ($\beta = 0.18$, $p = .033$) and the Algebra entrance exam in the base models ($\beta = 0.17$, $p = .050$). Altogether, the control measures explained between three and five percent of the variance for each outcome measure.

Stage 2: RPS acuity. RPS acuity was a significant predictor for each of the outcome measures ($\beta_{FractionComparison} = 0.38$, $\beta_{NumberLine} = -0.22$, $\beta_{FKA} = 0.43$, $\beta_{Algebra} = 0.34$, ps < .01). Introducing composite RPS acuity into each model more than doubled the variance explained for number line estimation and more than tripled the variance explained for the other three outcomes. Moreover, when RPS acuity was introduced, ANS acuity was no longer a significant predictor for any of the four outcomes. This indicates that much of the predictive power of ANS acuity in Step 1 was actually due to shared variance between ANS and RPS acuity. Squared semi-partial correlation coefficients (sr^2) indicate the unique variance explained by each of the variables in the final model over and above any shared variance among items. In the bottom rows of Tables 3 – 6, we have computed a ratio of the sr^2 for RPS acuity to the summed sr^2 of all control variables. This serves as an index of how large the independent unique explanatory power of RPS acuity was relative to the summed unique contributions of the other variables. For each of the outcome variables, the unique contribution of the RPS composite was 4.5 to 9.9 times larger than the summed unique contributions of the other three variables. Indeed, the effects of the control variables were small enough that the bivariate scatterplots of Fig. 4 are near-accurate indications of the overall relations between RPS acuity and the four outcome variables in the multiple regression framework.

Discussion

These results demonstrate that individual differences in sensitivity to nonsymbolic ratio magnitudes predict symbolic math competence. Participants with higher RPS acuity performed better on three symbolic measures of fractions knowledge and on a measure of algebra knowledge. This correlational evidence is consistent with our hypothesis that the ability to process nonsymbolic ratio magnitudes may be an important proto-mathematical ability upon which emerging mathematical competence is built. This has several potential implications for theories of human numerical cognition:

1. Humans have intuitive access to nonsymbolic ratio magnitudes that may support symbolic number knowledge. Jacob et al. (2012) previously made a similar point, yet this position remains novel. Many cognitive scientists have argued that fractions are not directly accessible to intuition, and education researchers have argued that sound conceptions of fractions must be founded on partitioning wholes into countable elements (Pitkethly & Hunting, 1996; Steffe, 2001). Hence, although visuospatial representations like pies and number lines are used extensively to teach fractions, they are typically used in contexts emphasizing counting as opposed to contexts attempting to leverage the RPS.

Why might RPS acuity explain differences in symbolic math outcomes despite this fact? Although concrete representations are typically used to elicit whole number concepts and procedures, RPS networks continue to encode holistic ratio magnitudes of these representations (Jacob & Nieder, 2009). Thus, educational experiences build symbolic-to-nonsymbolic links implicitly even when the RPS is not an explicit focus, helping to improve learners' understanding of symbolic fraction magnitudes. This mechanism is generally compatible with accounts of how perceptual learning supports mathematical cognition (e.g., Goldstone, Landy, & Son, 2010; Kellman, Massey, & Son, 2010).

This is not to suggest that the RPS alone is sufficient for building robust fractions knowledge. As noted by Kieren (1976), robust fractions knowledge requires mastery of several interpretations, including the important part-whole interpretation that relies heavily on partitioning and counting. However, that the RPS may provide a more intuitively accessible route for promoting the critical magnitude interpretation of fractions highlighted by recent research.

Why conclude that RPS abilities are antecedents of mathematical ability instead of vice versa? RPS acuity in trained rhesus monkeys approximates that of college undergraduates (Vallentin & Nieder, 2008) which may suggest it is a primitive ability, existing prior to instruction. However, there are no baseline studies of rhesus RPS prior to training. Ultimately, we can draw no firm conclusions about the relations between the RPS and education without additional investigation of the time course of RPS development and its relation to mathematical abilities.

2. The RPS potentially extends accounts of the nonsymbolic foundations of number to include not only fractions, but to include any number that can be presented as the ratio between two continuous quantities. Contrary to accounts that intuitive access to number ends at whole numbers, our results and others show that humans intuitively process magnitudes that correspond to nonsymbolic ratios like those made of line segments. Because line segments can be made arbitrarily long, such ratios can be made corresponding to any positive real number. The formal definition of pi is the ratio between a circle's circumference and its diameter. Unrolling a circumference and placing it next to the diameter yields same sort or ratio used in this experiment.

3. The RPS operates largely independently of the ANS as currently conceived. We demonstrated that the predictive power of RPS acuity was independent of–and far greater than–that of the ANS for four symbolic outcomes. Moreover, RPS acuity was measured effectively using ratios of line segments, even though line segments should not have engaged the ANS. Observed RPS effects also contrast with recent findings suggesting the link between ANS and symbolic math ability fades as children grow older (Fazio et al., 2014). On balance, it is clear that nonsymbolic RPS-based abilities are operative in multiple formats and cannot be considered simply subsidiary to the ANS.

4. Focusing on the RPS may help enrich theories of how symbolic numerical abilities develop. Currently, some researchers speculate that the ANS supports the acquisition of numerical concepts. Part of the argument's appeal lies in the fact that there is a one-to-one map between discrete numerosities and the counting numbers. However, it seems the same systems that estimate absolute magnitude (including the ANS) similarly process ratio magnitude (Jacob et al., 2012). To the extent that the RPS can be shown to interface with and influence symbolic number processing, it follows that the expansive psychophysical apparatus can influence numerical development. This possibility would firmly situate numerical development within the more generalized sense of magnitude as postulated by Walsh (2003). Future work investigating commonalities in general magnitude perception and explicitly numerical cognition will be necessary to delimit these relations and their practical implications.

Conclusion

Our findings indicate that individual differences in the RPS–a basic nonsymbolic processing capacity which until recently has received little study–predict outcomes on three different measures of symbolic fractions ability and algebra entrance exam performance at a major selective university. This stands in stark contrast to the assertion fractions concepts are unsupported by primitive architectures. Indeed, the RPS may prove to be an underappreciated neurocognitive startup tool that serves as a primitive ground for advanced mathematical concepts. With this in mind, we suggest that the time has come for the field to focus on sensitivity to ratio magnitudes: Instead of relegating ratio to the background as some parameter that influences comparison tasks, perhaps we should elevate ratio sensitivity to the foreground. After all, wouldn't it be poetic if math learning hinged on having a general sense of proportion?

Author Contributions

P.G. Matthews, M.R. Lewis and E.M. Hubbard all contributed to the study concept and design. M.R. Lewis programmed stimulus presentation software. Testing and data collection were performed by personnel from labs of P.G. Matthews and E.M. Hubbard, which included M.R. Lewis. M.R. Lewis and P.G. Matthews performed the data analysis and interpretation. P.G. Matthews, M.R. Lewis, and E.M. Hubbard drafted the manuscript. All authors approved the final version of the manuscript for submission.

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