Neurocognitive Architectures and the Nonsymbolic Foundations of Fractions Understanding

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Abstract

Children and adults experience pervasive difficulties understanding symbolic fractions. These difficulties have led some to propose that components of the human neurocognitive architecture, especially systems like the approximate number system (ANS), are ill-suited for learning fraction concepts. However, recent research in developmental psychology and neuroscience has revealed neurocognitive architectures – a “ratio processing system” (RPS) – tuned to the holistic magnitudes of nonsymbolic ratios that may be ideally suited for grounding fraction learning. We review this evidence alongside our own recent behavioral and brain imaging work demonstrating that nonsymbolic ratio perception is related to understanding fractions. We argue that this nonsymbolic RPS supports symbolic fraction understanding, similar to how the ANS supports whole number understanding. We then outline a number of open questions about the RPS and the ways in which it may be used to support fractions learning.

Keywords: fractions; neurocognitive architecture; educational neuroscience; fractions instruction; nonsymbolic ratios; ratio processing system; perceptually-based training

Introduction

Understanding fractions is critically important for the development of mathematical competence (National Mathematics Advisory Panel, 2008; Siegler, Fazio, Bailey, & Zhou, 2013). Fractions are a key component of elementary and middle school math curricula and serve as a foundational concept for algebra and higher mathematics (Booth & Newton, 2012). For instance, 10-year-olds' fraction knowledge predicts later high school algebra skills and overall math achievement (Siegler et al., 2012). Unfortunately, both children and adults – including teachers and highly educated professionals – struggle to understand fractions (Lipkus, Samsa, & Rimer, 2001; National Mathematics Advisory Panel, 2008; Newton, 2008; Post, Harel, Behr, & Lesh, 1991; Reyna & Brainerd, 2008; Stigler, Givvin, & Thompson, 2010).

Research in mathematics education and cognitive psychology has identified several common difficulties faced by learners regarding fractions. One of the most important is a difficulty accessing the holistic magnitudes of symbolic fractions (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Kallai & Tzelgov, 2009, 2012; Stafylidou & Vosniadou, 2004) – a difficulty that even expert mathematicians face (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013). Understanding fraction magnitudes is an important piece of conceptual knowledge that predicts accuracy on fraction computation, estimation, and word problems (Hecht, Close, & Santisi, 2003; Hecht & Vagi, 2010; Hecht, 1998; Siegler et al., 2013; Siegler & Pyke, 2013). Perhaps more importantly, understanding the magnitude of symbolic fractions predicts general mathematics achievement (Bailey, Hoard, Nugent, & Geary, 2012; Siegler & Pyke, 2013; Siegler, Thompson, & Schneider, 2011). Unfortunately, misunderstandings of fraction magnitude are all too common. For example, Mack (1990) found that typical 6th graders often claimed that 1/8 is greater than 1/6. Similarly, when a nationally representative sample of 8th graders was asked whether 12/13 + 7/8 was closest to 1, 2, 19, or 21, more chose 19 and 21 than 2 (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981). Outside the classroom, the failure to understand fraction magnitude even affects fast food choices, with as many as half the participants in a consumer focus group reporting that they thought the A&W 1/3 pound burger was smaller than McDonald's famous "quarter pounder." They therefore chose the smaller McDonald's burger, even though they preferred the taste of the A&W burger (Taubman, 2007 p. 62-63). This real-world example illustrates a clear failure to recognize that the magnitude of a fraction emerges from the relation between its two components (i.e., the numerator and denominator) rather than from the magnitudes of the individual components themselves.

A second common misunderstanding, known as "the whole number bias", is closely linked to the first. This bias is so named because students' struggles with fractions often reflect the misapplication of concepts and procedures that are appropriate for whole numbers but inappropriate for fractions (see Ni & Zhou, 2005 for a review). This bias is reflected not only in the kinds of computational errors people make, but also in their verbal reports. For example, one child in Mack's (1990) study described fraction addition as follows: "Well, you go across. You add the top numbers together and the bottom numbers together" (p. 23).

A third common misunderstanding is the failure to recognize that fractions have different properties than whole numbers (Siegler et al., 2013; Vamvakoussi & Vosniadou, 2010). To name a few of these properties: a) fractions, unlike whole numbers, have no direct successor; b) there are an infinite number of fractions between any two numbers; and c) multiplying by a positive fraction sometimes produces a result that is smaller than the original operands, while dividing sometimes produces a result that is larger, depending on the magnitude of the multiplier.

Each of these misunderstandings can pose serious impediments for everything from acquiring a sense of fraction magnitude to acquiring a robust understanding of how arithmetic with fractions works. Designing effective instructional approaches to rectify such common misunderstandings – or to prevent them from arising - would prove fruitful for promoting robust fraction knowledge. In this chapter, we suggest that research into the perceptually-based capacities of the human cognitive architecture for nonsymbolic ratios, and teaching methods that build on these neurocognitive systems, may help provide a pathway for the amelioration and prevention of these common problems.

Fundamental Limitations of the Human Cognitive Architecture?

The pervasive problems with learning fractions have led some researchers to claim that fractions, unlike whole numbers, are largely incompatible with the human cognitive architecture (e.g., Bonato et al., 2007; Dehaene, 1997; Feigenson, Dehaene, & Spelke, 2004; Gallistel & Gelman, 1992; Geary, 2007; Gelman & Williams, 1998). According to these “innate constraint” accounts (Ni & Zhou, 2005), whole number symbols and concepts are relatively easy for most children to learn because they are supported by core neurocognitive representational systems with long phylogenetic histories (Feigenson et al., 2004; Piazza, 2010). One proposed core system is the approximate number system (ANS). The ANS, situated in the intraparietal sulcus (IPS), endows humans – and indeed many nonhuman animals – with the ability to represent the approximate number of discrete elements in a set (Brannon & Roitman, 2003; Dehaene, Dehaene-Lambertz, & Cohen, 1998; Piazza, 2010; see Volume 1 of this book series). It is widely hypothesized that early learning of whole number concepts builds upon this system (e.g., Dehaene, 1996; Feigenson et al., 2004; Nieder, 2005; Piazza, 2010). Innate constraints theorists go on to argue that the ANS – although naturally suited to whole numbers – is poorly suited for supporting the representation, conceptualization and knowledge of fraction magnitudes.

In contrast to this pessimistic view, emerging evidence suggests that the human cognitive architecture may contain systems ideally suited for supporting fraction learning. In fact, there may exist neurocognitive startup tools (Piazza, 2010) that can be exploited to cultivate a robust understanding of key fraction concepts, like magnitude, and to help decrease the prevalence of common difficulties. On our view, many misunderstandings of fractions stem not from a fundamental incompatibility between fractions concepts and the human cognitive architecture but rather because current teaching methods do not effectively recruit and leverage our existing perceptual abilities. In particular, similar to Sielger et al. (2011; 2013), we suggest that teaching methods that fail to build an understanding of fractions as magnitudes lead to the conceptual misunderstandings noted above, and to rote (mis)application of procedures for manipulating fractions. Our account complements Siegler et al. by suggesting that nonsymbolic ratio processing might be a critical neurocognitive primitive for understanding fractions as magnitudes and by exploring how these primitives are linked to systems for symbolic fraction understanding.

Below, we connect the evidence, currently scattered across neuroscience, psychology, and education research, to lay out an agenda for developing an educational neuroscience of a ratio processing system (RPS), similar to the ANS that may provide a nonsymbolic foundation for fraction understanding. This account may eventually deepen our understanding of this fundamental component of mathematical cognition and inform educational practice.

A Competing View: The Ratio Processing System

A growing body of evidence demonstrates that humans and nonhuman primates possess a ratio processing system (RPS), a set of neurocognitive architectures that support the representation and processing of nonsymbolic ratios. Indeed, sensitivity to the magnitudes of nonsymbolic ratios has been demonstrated in non-human primates (Vallentin & Nieder, 2008, 2010; Woodruff & Premack, 1981), pre-verbal infants (McCrink & Wynn, 2007), elementary school-age children (Boyer, Levine, & Huttenlocher, 2008; Duffy, Huttenlocher, & Levine, 2005; Meert, Gregoire, Seron, & Noël, 2013; Sophian, 2000; Spinillo & Bryant, 1999), typically developing adults (Hollands & Dyre, 2000; Meert, Grégoire, Seron, & Noël, 2011; Stevens & Galanter, 1957), and individuals with limited number vocabularies and formal arithmetic skills (McCrink, Spelke, Dehaene, & Pica, 2013). This widespread competence – even among innumerate subjects and nonhuman animals – demonstrates that these architectures are present even in the absence of formal education and likely predates the emergence of *Homo sapiens* (Jacob, Vallentin, & Nieder, 2012).

Although symbolic fractions are not usually formally introduced until grades 3 and 4 (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), children far younger than this demonstrate some understanding of nonsymbolic ratios (Jeong, Levine, & Huttenlocher, 2007; Sophian, 2000; Spinillo & Bryant, 1991, 1999). McCrink & Wynn (McCrink & Wynn, 2007) demonstrated that 6-month old infants who were habituated to specific nonsymbolic ratios (e.g., a 2:1 ratio composed of yellow Pacmen intermixed with blue pellets) subsequently looked longer at novel ratio stimuli that differed by a factor of two (e.g., a 4:1 ratio). By 4 years of age, children can order pictures of part-whole ratios based on magnitude (Goswami, 1995) and perform above chance on tasks that require the addition and subtraction of nonsymbolic part-whole ratios (Mix, Levine, & Huttenlocher, 1999) even though much older children struggle to perform analogous tasks with symbolic fractions (Mack, 1990). Moreover, this competence with nonsymbolic ratios seems to be flexible and abstract, allowing children to draw proportional analogies even when materials are visually dissimilar (e.g., drawing analogies between pizzas and chocolates and glasses of lemonade) (Goswami, 1989; Singer-Freeman & Goswami, 2001).

Recent findings suggest this early sensitivity to nonsymbolic ratios may be supported by fronto-parietal circuitry dedicated to the processing of this kind of information. Neuroimaging studies of adult humans and single-unit physiology studies of non-human primates have revealed a fronto-parietal network that is sensitive to the holistic magnitudes of nonsymbolic ratios (see Figure 1). Evidence using fMRI-adaptation methods (Grill-Spector, Henson, & Martin, 2006; Grill-Spector & Malach, 2001; Krekelberg, Boynton, & van Wezel, 2006) with humans provides indirect evidence of neuronal tuning to specific ratios (Jacob & Nieder, 2009b). When humans are repeatedly presented with the same nonsymbolic ratio (composed of dots or lines), populations of neurons in the prefrontal cortex and parietal cortex near the IPS adapt to that ratio, resulting in lower blood oxygen level-dependent (BOLD) signals. Presentation of a novel ratio activates a different population of neurons, leading to recovery from adaptation, reflected in stronger BOLD signals. The strength of this recovery is dependent on the holistic distance between the novel ratio and the adapted ratio (see values shown in Figure 1). Similarly, direct recordings from rhesus monkeys (*Macaca mulatta*) trained to perform delayed match-to-sample tasks with nonsymbolic line ratios have shown that some neurons in monkey prefrontal and parietal regions are selectively tuned to specific visuospatial ratios (Vallentin & Nieder, 2010). Within this network, subpopulations of neurons show classical distance-dependent responses to particular ratios. For example, a population of neurons tuned to 3/5 will activate strongly in response to 3/5, less strongly for close but non-identical ratios (e.g., 4/7 or 5/8), and weakly for more distant ratios (e.g., 1/4 or 8/9). Notably, this tuning is not sensitive to the absolute sizes of components used to instantiate a ratio but instead tracks the holistic magnitude corresponding to the relationship between two given components.

INSERT FIGURE 1 ABOUT HERE

How the RPS May Influence Fraction Learning

The findings reviewed above suggest that the RPS may exist as a phylogenetically ancient system that processes ratios of objects, complementary to the ANS with its role in supporting whole number processing. Hence, neurocognitive architectures tuned to the magnitudes of nonsymbolic ratios—such as the relative lengths of two lines or the relative areas of two figures—are present even before learners receive fraction instruction. On this account, children are equipped with cognitive architectures that support ratio magnitude concepts prior to formal education in the same way that their innate ability to estimate approximate numerosities (sets of countable objects) equips them to learn about whole numbers (e.g., Nieder & Dehaene, 2009). We further propose that leveraging perceptual sensitivity to nonsymbolic ratio magnitudes can play a major role in fostering the developing an understanding of symbolic fractions as magnitudes, which in turn will improve both conceptual and procedural fraction processing. We argue that making explicit connections between symbolic fractions and RPS representations of nonsymbolic ratio magnitude may prove to be a key step for promoting successful fraction learning.

INSERT FIGURE 2 ABOUT HERE

Figure 2 presents our hypothesized model of how the RPS may influence learners' understanding of fractions. Within this model, we propose that guided experience with symbolic fractions in concert with nonsymbolic instantiations – both in and out of school – can help generate links between analog RPS based representations of magnitudes and fraction symbols, whether they be verbal labels like "one third" and "one half" or orthographic symbols like "2/3" and "3/4". In this way, the RPS can help provide an intuitive grounding for understanding fractions symbol as representing magnitudes. We hypothesize that the precision of these links is moderated by a learner's RPS acuity: higher acuity provides a more precise representation of magnitude, furnishing a more reliable grounding relative to the representations provided by lower acuity.

Note that according to this hypothetical model, these mechanisms should operate even when instruction does not explicitly attempt to capitalize on RPS abilities, albeit suboptimally. Visuospatial representations like pies, number lines and counters are used extensively to teach about fractions, typically in the context of a pedagogy that emphasizes both counting and partitioning. Even though counting and partitioning elicit whole number concepts and procedures, RPS networks—which antedate formal educational experiences—still encode the holistic ratio magnitudes of these visuospatial representations. In this way, educational experiences continue to build symbol-to-percept links in an implicit fashion even when the RPS is not an explicit focus. Over time, these links strengthen, providing intuitive knowledge of fraction magnitudes, precisely the sort of magnitude knowledge which Siegler et al (2011) have shown to be predictive of fractions arithmetic proficiency and overall scores on mathematics achievement tests. However, when such instruction focuses too early on strategies that involve partitioning and counting, learners may activate incompatible ANS representations and whole-number schemas, leading to less effective learning. We argue that instruction that seeks to optimally leverage the continuous representations of nonsymbolic ratio magnitude conferred by the RPS can make the symbol-percept links stronger and at earlier points in development.

Note that our model does not assume that this sort of perceptual training alone will result in developing fractions mastery. It is important to recall that it takes years for young children to gain explicit knowledge of the cardinal values of natural numbers even after memorizing the count sequence (e.g., Briars & Siegler, 1984; Le Corre & Carey, 2007, Piazza, 2010). Given the inherent complexity of fractions concepts, our hypothesis should not be taken to support massive and discontinuous reorganization of knowledge structures; instead, we hypothesize that symbol-percept links can form an important basis of a learner's understanding of the holistic magnitude of fractions. Grounding the meaning of fractions symbols in intuitively accessible representations of magnitude can help provide a key understanding that promotes the acquisition of procedural and conceptual fraction knowledge. Because fraction knowledge has been found to be a critical gatekeeper of subsequent algebra achievement (Booth & Newton, 2012; Siegler, Duncan et al, 2012), this model also suggests that the quality of these links and the fraction understanding they support could have downstream effects on students' algebra learning.

Three key predictions flow from the RPS model. First, the model suggests that individual differences in the functioning of nonsymbolic ratio circuits should predict individual differences in symbolic fraction competence, similar to studies demonstrating that individual differences in the acuity of the ANS predict math achievement (Halberda, Mazzocco, & Feigenson, 2008; Libertus, Feigenson, & Halberda, 2011; Mazzocco, Feigenson, & Halberda, 2011; but see de Smedt, Noel, Gilmore & Ansari, 2013) and dyscalculia (Mazzocco, Feigenson, & Halberda, 2011; Mussolin, Mejias, & Noël, 2010; Piazza et al., 2010; Price, Holloway, Räsänen, Vesterinen, & Ansari, 2007). However, even when ANS acuity correlates with math skills, it does so less strongly than do symbolic number skills (e.g., Chu et al. 2013; Lyons et al., 2014), suggesting that symbolic representations and the strength of the links between the ANS and symbolic representations are important for predicting mathematical ability (see de Smedt et al., 2013 for a review). Similarly, our model focuses not only on RPS acuity, but also on the strength of the links between the RPS and symbolic fractions. To date no studies have explored whether individual differences in ratio processing acuity can predict either typical mathematical achievement or learning disability.

Second, the model suggests that the strength and accuracy of the symbolic-nonsymbolic links that learners generate are influenced by education/experience and that the quality of these links may be amenable to targeted interventions. One reason why students may not be generating strong links between symbolic and intuitive nonsymbolic representations of fraction magnitude is that most fraction curricula place an emphasis on activities like partitioning that may activate incompatible ANS representations and whole-number schemas. One way to rectify this situation and to improve learners' links may be to exploit perceptual learning paradigms (see Figure 4) to help learners associate fraction symbols with relational magnitudes rather than with discrete numerosities.

Finally, at the neural level, this model suggests that RPS circuits may be involved in the processing of both nonsymbolic ratios and symbolic fractions for adults. A handful of neuroimaging studies suggest that the same fronto-parietal cortical networks implicated in the representation of nonsymbolic ratios are also involved in representing and processing symbolic fractions. Fronto-parietal activation patterns are observed during symbolic fraction addition and subtraction (Schmithorst & Brown, 2004). Even simple comparison of symbolic fractions activates a network of fronto-parietal areas, with distance-dependent activation in a right IPS area (Ischebeck, Schocke, & Delazer, 2009) similar to that activated during nonsymbolic ratio tasks. Finally, repeated presentation of symbolic fractions (Arabic numerals and fraction words) leads to adaptation and distance-dependent recovery in frontal and parietal regions (Jacob & Nieder, 2009a), showing that these networks are recruited even in the absence of specific task demands. Despite this tantalizing evidence, no studies have directly investigated whether the representation and processing of symbolic fractions directly engage nonsymbolic RPS circuitry. For example, when adults represent the symbolic fraction 1/4, does this engage populations of neurons that are tuned to a nonsymbolic ratio of equivalent value? Of course, this also means that no studies have studied this from a developmental perspective. That is, even if 1/4 engages RPS circuitry, how does this engagement change with age and experience?

Emerging Behavioral and Neuroimaging Evidence for RPS Model Predictions

We are currently investigating several key predictions of the RPS model using a combination of behavioral methods, targeted training, and fMRI. In one strand of research, we are investigating whether individual differences in nonsymbolic ratio acuity predict individual differences in symbolic fraction knowledge and algebra achievement. In a second strand, we are investigating whether the quality of learners' symbolic-to-nonsymbolic links is amenable to intervention and whether improving these links will improve performance on a measure of fraction knowledge. In a third strand, we are using an fMRI adaptation paradigm (see Figure 1 above and Study 3 below) to test whether adults' representations of symbolic fractions recruit the same circuitry as their representations of nonsymbolic ratio magnitudes. We summarize the results of three studies representing each of these strands below.

Study 1 – Correlations between RPS acuity and Fraction Knowledge.

To investigate whether individual differences in RPS acuity predict fraction knowledge and subsequent algebra achievement, we have constructed four novel measures of RPS acuity (see Figure 3) similar those used to measure ANS acuity. In each of these tasks, the participant's task was simply to choose the larger of two nonsymbolic ratios. In addition to this battery of RPS tasks, participants completed an array of cognitive tasks, including: 1) Three measures of symbolic fraction knowledge (symbolic fraction comparison, number-line estimation with fraction stimuli, and a paper-and-pencil fraction knowledge assessment). 2) Two tasks to control for differences in participants' acuity for processing nonsymbolic ratio components (a standard ANS acuity task and a line length discrimination task). 3) A flanker task[1] to control for differences in inhibitory control that may affect participants' ability to suppress componential processing when evaluating the magnitudes of ratio stimuli. Finally, we requested access to participants' algebra subtest scores from their university entrance examinations.

INSERT FIGURE 3 ABOUT HERE

The key research question for Study 1 was whether RPS acuity predicts symbolic fraction knowledge and algebra achievement above and beyond ANS acuity and other control measures. To date, 178 undergraduate students have completed the study, and the results have been consistent with model predictions. For example, the ability to successfully compare the

[1] The flanker task is a classic test of inhibitory control, in which participants are asked to indicate the direction in which a central arrow points while ignoring an array of "flanking" arrows. The difference between reaction times for congruent trials (e.g. picking "right" for >>>>>>) and incongruent trials (e.g., "right" for <<><<) is the key measure.

magnitudes of nonsymbolic line ratios predicts symbolic fraction comparison ability (see Figure 4), even after accounting for performance on control tasks. We have found that RPS acuity similarly predicts number-line estimation ability and performance on the fraction knowledge assessment. We are currently developing match-to-sample protocols that can be used to test for the same correlations among school-age children. These protocols are less dependent on language, which will better permit us to investigate the development of nonsymbolic ratio discrimination over time and its connection to fraction learning. This developmental question will ultimately prove to be a critical test for our theory.

INSERT FIGURE 4 ABOUT HERE

Study 2 - Training Symbolic-Nonsymbolic Links

Whereas Study 1 provides evidence that the acuity of a learner's RPS is significantly related to fraction learning, the correlational nature of the evidence does not allow us to address questions of causality. Additionally, Study 1 does not directly assess the quality of the symbolic-to-nonsymbolic links that are a key feature of our hypothesis regarding how RPS acuity could affect symbolic fraction knowledge. In Study 2, we are exploring these questions by testing a perceptually-based training paradigm in which participants improve the quality of their symbolic-nonsymbolic links by learning to associate the magnitude of nonsymbolic ratios with corresponding symbolic fractions of the same value. Our initial pilot studies used three 20-minute computerized training sessions spread across three days in which participants learned these associations by completing either cross format match-to-sample tasks with feedback or by completing cross format comparison tasks with feedback (see Figure 5 for an illustration). These tasks were adaptive: as participants improved in accuracy, the program reduced the distance between target and distractor ratios, with the goal of increasing the precision of participants' links between symbolic fractions and their corresponding nonsymbolic ratio magnitudes.

The pilot data shown in Figure 5 were collected from eight graduate students, and the results have been encouraging despite the small size of our initial sample. These highly educated students showed statistically significant gains in performance on symbolic fraction comparison tasks as a result of the training, suggesting that improving the links between nonsymbolic and symbolic representations can indeed improve processing of symbolic fractions. We are currently running a full version of this experiment with undergraduate students and plan to use these results to inform similar training interventions with school-age children.

INSERT FIGURE 5 ABOUT HERE

Study 3 - fMRI adaptation

In Study 3, we are investigating the neural signature of these symbolic-to-nonsymbolic links. Specifically, we are investigating whether links between representations of nonsymbolic ratio magnitudes and symbolic fraction representations result in the same sort of distance-dependent recovery from adaptation (Figure 6) that has been observed between nonsymbolic numerosity (dot arrays) and symbolic whole numbers (Piazza, Pinel, Le Bihan, & Dehaene, 2007). To date, six adults and one ten-year old child have completed this study. During the study, participants were repeatedly presented with a specific nonsymbolic ratio magnitude. In one run, participants viewed a series of line ratios in which the component line lengths varied but the shorter line was always 3/10 as long as the longer line, while in the other, the adapting ratios were always 7/10. During each run, subjects were also presented with 15 deviant magnitudes, presented in each of two notations (symbolic, nonsymbolic) and distances (near, far). For example, after presenting a stream of 7/10 stimuli, a near symbolic deviant might be the fraction 7/9 and a far symbolic deviant might be the fraction 1/3. Likewise, after presenting a stream of 3/10 stimuli a near nonsymbolic deviant might be a line ratio in which the shorter line is 1/4 as long as the longer line; and a far nonsymbolic deviant might be a line ratio in which the shorter line is 5/7 as long as the longer line (see Figure 6).

INSERT FIGURE 6 ABOUT HERE

Providing initial confirmation of the model predictions, we find the expected distance effect, with larger responses to far deviants than to close deviants, in the right mid-IPS (Figure 7). This is true both when we analyze the data collapsed across line ratios and symbolic fractions (Figure 7A) and when we examine the critical cross-notation rebound effect by restricting our analyses to digits (Figure 7B). The fact that adaptation transfers from nonsymbolic line ratios to symbolic fractions in a distance-dependent manner suggests that adults have made the links between symbolic fractions and the more basic RPS system that represents fraction magitude, as predicted by our model. These links are apparent even in the absence of explicit task demands to map between different dimensions (indeed, participants were performing a completely irrelevant task in which they merely detected the brief dimming of the fixation cross), which is considered to be a key method for isolating representations (see Cohen Kadosh & Walsh, 2009 and commentaries contained therein). Building on this basic foundation, we will begin to explore individal differences in RPS sensitivity and the strength of the links between the RPS system and fractions, in adults and children. In particular, it will be critical to examine developmental changes in these links as a result of formal instruction in fractions, by comparing, for example, 2nd, 5th and 8th graders, who are at different stages of fraction instruction.

INSERT FIGURE 7 ABOUT HERE

Open Questions

Although our work reviewed above has begun to delimit the nature of the RPS and its relationship with symbolic fractions processing, much more remains to be done. We believe that the RPS may represent a core numerical competence that has yet to be optimally leveraged to support fraction learning. However, a number of key questions need to be answered in order to build a scientific understanding of this ability that both deepens our understanding of mathematical cognition and produces knowledge that can inform fraction instruction. Our first set of questions involves basic neuroscientific inquiry into the phylogenesis and ontogenesis of the brain's ratio processing system and its contribution and connectivity to other brain networks. A second set of questions involves the potential role of the RPS in both individual differences in mathematical abilities and dyscalculia. A third set of questions involves developing and testing a host of ways to improve fraction learning by more directly leveraging the RPS thereby imbuing symbolic fractions with meaning. Finally, addressing these first three questions, allows us to ask a fourth, more nuanced, set of questions. When in symbolic fractions learning is the RPS most important? Are its effects primarily limited to helping young children acquire basic understandings of fraction magnitude, or can leveraging the RPS even help more advanced students improve their proficiency with mathematical operations on fractions?

Charting the development and architecture of the RPS

Although there is substantial neuroimaging data on the development of parietal cortex generally (for a recent review, see Giedd et al., 2015) and the development of the ANS (e.g., Cantlon et al., 2006; Izard, Dehaene-Lambertz, & Dehaene, 2008) there is no developmental neuroimaging evidence regarding the ontogenesis of the RPS. The only neuroscientific evidence regarding the brain's representation of nonsymbolic ratios comes from fMRI studies of adult humans and single-unit studies of trained monkeys (Jacob & Nieder, 2009b; Vallentin & Nieder, 2008, 2010). Charting the brain's ability to represent nonsymbolic ratios prior to, during, and after formal education on fractions is a critical step in not only understanding how these architectures prepare children to learn about symbolic fractions but also in illuminating how these architectures may themselves be reshaped by formal education.

Another key question regards how RPS architectures relate to systems that are tuned to the magnitudes of the components that create a nonsymbolic ratio (e.g., the length of the two lines in a line-ratio or the numerosity of the two sets of dots in a dot-ratio). Our current studies statistically control for the effects of these component processing systems, such as the ANS, but do not directly investigate pathways relating the two. Distance-dependent activation for component dimensions such as physical size and numerosity are observed in IPS regions (Cohen Kadosh et al., 2005; Pinel, Piazza, Le Bihan, & Dehaene, 2004) near areas associated with the processing of nonsymbolic ratios, raising the possibility that the RPS is a higher order extension of one or more of these representational systems. Determining how these systems interact may help to illuminate early differences in children's understanding of continuous and discrete ratios

(Boyer et al., 2008; Jeong et al., 2007; Spinillo & Bryant, 1999) and may also help to inform the long-standing educational debate about the relative efficacy of different pedagogical representations of fractions (Cramer & Wyberg, 2009). Arguments for using different types of representations like number lines, area models, or counters are usually justified by appealing to higher level ratio number schemas like measurement, partitioning, or equal sharing, but the choice of fraction representations may also be informed by lower level architectural considerations. For example, if the RPS is robustly connected to systems tuned to continuous magnitudes but not to systems tuned to discrete magnitudes, it may be most effectively leveraged by using continuous as opposed to discrete stimuli.

Leveraging the RPS to support fraction learning

There is good reason to believe that conventional fraction instruction fails to effectively leverage the brain's ability to represent nonsymbolic fractional magnitudes. The first stages of conventional fraction instruction usually rely on partitioning or equal-sharing (Ni & Zhou, 2005; Pitkethly & Hunting, 1996; Siegler et al., 2010). Such approaches involve dividing a figure into equal parts or partitioning a set of items (e.g., partitioning 12 candies into four equally numerous groups) and often fundamentally rely on counting. To identify the fraction illustrated by a partially shaded figure, a child counts the number of shaded parts, assigns this to the numerator and counts the total number of parts and assigns this to the denominator (Davydov & Tsvetkovich, 1991). As a result of this reliance on counting, the early stages of conventional fraction training may encourage the re-appropriation of count-based, whole number schemas rather than harnessing the capabilities of the RPS. This re-appropriation can have serious consequences (Mack, 1990; Ni & Zhou, 2005; Siegler et al., 2013). Indeed, Mack (1995) found that partitioning approaches often led third- and fourth-grade students to overgeneralize their whole number knowledge to fractions. This overgeneralization prevents children from grasping fraction concepts and can lead to common procedural errors such as saying that 12/13 + 7/8 is closer to 19 and 21 than to 2 (Carpenter et al., 1981).

We propose that fraction education may be improved by designing instruction that more directly leverages the RPS while reducing the misapplication of whole-number schemas. Following Feigenson et al. (2004), we argue that acquiring number concepts is easy when they are supported by core systems of representation and hard when this acquisition goes beyond the limits of a core system. However, we disagree with their conclusion that core number systems are incompatible with rational number concepts. Instead, we argue that ratio brain architectures might naturally support fraction concepts that the ANS cannot. Explicitly leveraging the RPS may help discourage the misapplication of whole number concepts and build a more generative foundation for future learning than conventional instruction.

We are not necessarily proposing a complete reformulation of fraction instruction, but rather offering a new theoretical basis for 1) imagining how fraction education can be better grounded in children's pre-existing abilities, and 2) systematically developing and testing modifications to conventional fractions instruction. Educators have long used nonsymbolic referents to teach fractions, justifying their use as attempts to ground understanding in children's informal knowledge (e.g., their knowledge of sharing) (Mack, 1990; Siegler et al., 2010) or to better illustrate the formal logic of rational number mathematics (Davydov & Tsvetkovich, 1991; Moss & Case, 1999; Wu, 2008). Here, we argue that an alternative way of conceptualizing early fraction education is as a process of building upon children's pre-existing abilities to perceive and represent magnitudes corresponding to nonsymbolic ratios. From this perspective, fraction learning does not need to start from scratch or require onerous abstraction; instead, fraction learning can build upon the solid foundation provided by nonsymbolic RPS architectures.

In practice, changes emerging from this perspective may appear small, but their impact may be profound. For example, this perspective suggests that it may be fruitful to replace or supplement conventional nonsymbolic referents composed of discrete, countable elements (e.g., pies and wedges or small sets) with uncountable nonsymbolic ratios like pairs of lines or uncountable sets of dots. Because these types of ratios are inherently uncountable, their use should help prevent the inappropriate application of whole-number knowledge (see also, Boyer et al., 2008; Jeong et al., 2007). Other changes might include the adoption of targeted interventions like the training paradigm we are employing in Experiment 2. The take home message is clear: if the brain of the elementary school child – like that of the human adult or the non-human primate – is able to represent the holistic magnitudes of these nonsymbolic ratios, pedagogies based on this capacity may also help children build an intuitive understanding of fraction magnitude that serves as a generative foundation for future learning.

We refer to a 'generative foundation' because the conceptual foundation built by leveraging the RPS may have effects far beyond cultivating an intuitive understanding of fraction magnitude. Building a stronger, more grounded understanding of fractions in the early stages of learning can support future learning of fractions and related concepts (e.g., decimals, percent, and measure) throughout elementary and middle school and can help prepare students for algebra (Booth & Newton, 2012; Siegler et al., 2012). Building a stronger foundation can also facilitate the profound reorganization of numerical reasoning that Siegler et al. (2011; 2013) have attributed to the acquisition of fraction concepts. In comparison to count-based methods, which may bind thought in terms of whole numbers, proper engagement of the RPS may potentially help students develop a clearer understanding of the relational properties of numbers, enabling them to better grasp key mathematical and scientific concepts like ratio, rate, and probability. Of course, at this early point, these intriguing possibilities remain speculative and stand in need of experimental verification before they can guide classroom practices. It is our hope that the current chapter will spur such experimental work.

RPS and Dyscalculia?

If the RPS does indeed serve as a core system upon which mathematical concepts and learning are built, deficits in nonsymbolic ratio acuity at the behavioral and neural levels might provide deeper explanations of atypical math achievement including developmental dyscalculia. Developmental dyscalculia is a severe difficulty in learning mathematics that affects 5-7% of the population (Butterworth, Varma, & Laurillard, 2011; Butterworth, 2005). One potential source of dyscalculia is a deficit in the ANS characterized by reduced nonsymbolic acuity. Some researchers have found that children with dyscalculia are often less accurate than IQ-matched peers in performing simple nonsymbolic comparison tasks like choosing the more numerous set of dots (Mazzocco et al., 2011; Piazza et al., 2010, for contrasting accounts see Gilmore et al, 2013; Szucs, Devine, Soltesz, Nobes & Gabriel, 2013). These behaviorally measured deficits are often observed in parallel at the neural level, with dyscalculic children showing reduced IPS activation during comparison tasks (Kucian et al., 2011; Price et al., 2007). If RPS deficits are also associated with dyscalculia, deficits in nonsymbolic ratio acuity at the behavioral and neural levels may also differentiate children with dyscalculia from their typically developing peers.

RPS deficits may be comorbid with ANS deficits. Alternatively, the two deficits may dissociate, resulting in distinct subtypes of dyscalculia. Under this scenario, children with both deficits might experience the most profound difficulties, children with ANS deficits alone might exhibit specific difficulty learning whole number facts and procedures[2], and children with RPS deficits alone might succeed in early whole number learning but struggle when they encounter fractions and higher levels of mathematics that depends on rational number concepts. This pattern would suggest that the group with ANS deficits alone might still be able to learn about fractions, but not through traditional approaches that stress whole-number concepts and use discrete referents that rely on the ANS. For the other two groups, it would suggest that they might need even more remediation when learning fraction concepts. Finally, if the two systems are dissociated, it raises the possibility that deficits in one system might be remediated in part by relying more heavily on the remaining functional system. For instance, children with impairments specific to the ANS might be more efficiently taught about number magnitudes using ratio representations of whole number concepts as opposed to using discrete whole number representations.

Summary

Recent research has revealed a set of neurocognitive architectures – the ratio processing system or RPS – that are tuned to the holistic magnitudes of nonsymbolic ratios. We have proposed that the ability to represent ratio/fraction magnitudes conferred by the RPS may support a better understanding of fractions as relative magnitudes, and may even help children to achieve the conceptual insight that fractions represent magnitudes. Specifically, we hypothesize that learning can be improved by increasing the degree to which instruction leverages this brain network, helping ground children's understanding of symbolic fractions in their preexisting abilities to represent holistic ratio magnitudes intuitively. This improved understanding of

[2] It is important to note that whole number deficits are related at least in part to retrieving math facts from long-term memory (Geary, 1993) and this in turn may be related to frontal-hippocampal problems (e.g., Qin et al., 2014). Hence, math learning difficulties, at both a behavioral and anatomical level are likely to be more complex than just ANS difficulties. Similar complexity is also likely to be present for fractions learning.

fractions as magnitudes, in turn, will support better conceptual understanding of fractions and more appropriate application of arithmetic procedures with fractions. A number of key questions need to be answered in order to build an educationally relevant model of the RPS that is sufficiently detailed to deepen our understanding of mathematical cognition and to usefully inform real-world educational interventions. Despite these open questions, compelling evidence is accumulating that the RPS does indeed exist, and is another example of what Piazza (2010) calls a neurocognitive startup tool for supporting mathematical cognition. Early research from our lab suggests that the RPS may be critically important for learning about fractions. Future work delimiting its function and its engagement during symbolic fraction learning holds significant promise for enriching our understanding of human numerical cognition.

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